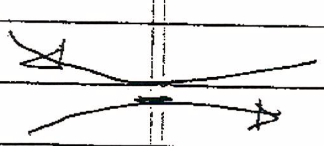
Simple Ideas in Non-Ideal MHD I

→ Freezing-in law:

$$\frac{d}{dt}\underline{B} = \underline{B}\cdot\nabla\underline{v} + \eta\nabla^2\underline{B}$$

↑ breaking → small scale → singularity / turbulence

→



→ ~~current sheet~~ singular layers

→ sites of reconnection → boundary layer problem.

→ topology change

→ so

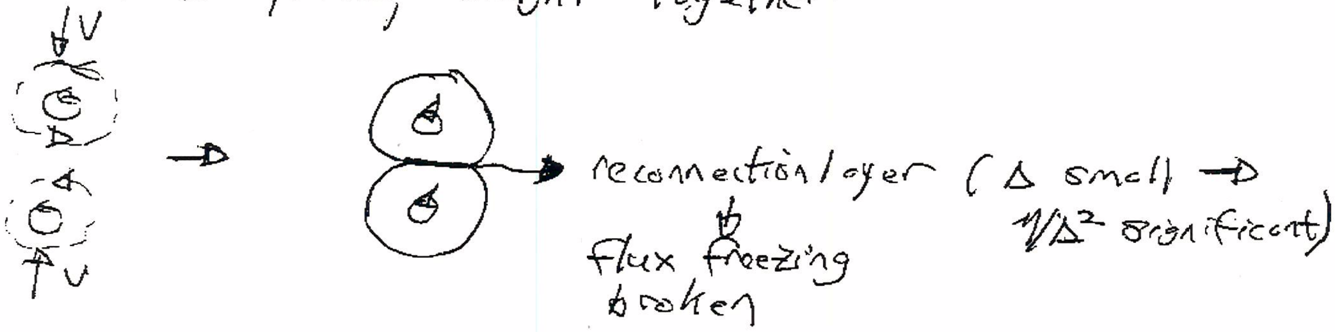
- Sweet-Parker Reconnection theory
- Re-visiting magnetic helicity; Taylor Theory → more coming
- anomalous resistivity (again)
- flux expulsion

↓ next

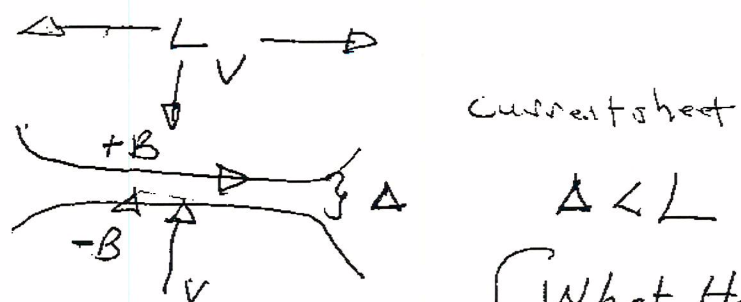
→ Breakdown of Flux Freezing – Magnetic Reconnection ?!

Simple Example: Sweet – Parker Problem
(re-visit later)

→ consider two cylinders of plasma, carrying current ⊥ plane, brought together



⇒ consider layer



2 plasma slabs brought together at V

$\Delta < L$

What Happens?
Stationary Solution Possible?

$\nabla \cdot \underline{V} = 0$

$$\frac{d}{dt}\underline{B} = \underline{B} \cdot \underline{\nabla}\,\underline{V} + \eta \nabla^2 \underline{B}$$

$$S_{ij}^{(0)} = \begin{pmatrix} 0, & 0 \\ 0, & -V\delta(y) \end{pmatrix}$$

$\underset{\sim}{S}$ rate-of-strain tensor

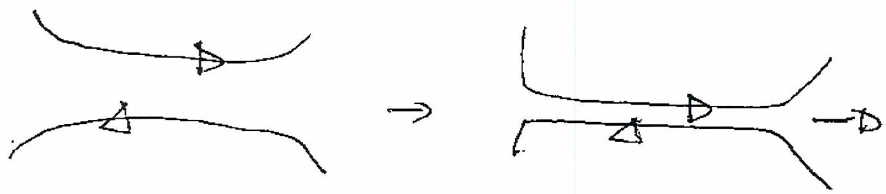
→ singularity!

tip-off of small scale generation in $\underline{B}$
⇒ resistive diffusion, breaking of freezing-in . . .

i.e. for stationary solution,

$$-\frac{\underline{B}\cdot\underline{\nabla}\,\underline{V}}{\eta} = \nabla^2 \underline{B}, \qquad \underline{\nabla}\cdot\underline{V} = 0$$

$\Rightarrow$

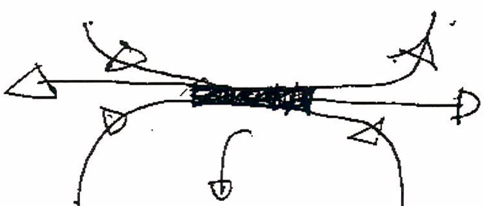


$\rightarrow$ Δ small

$\nabla(B^2/8\pi)$ increasing

fluid expelled,

$\Rightarrow$



reconnection region

out flow along B_0

expect opposed fields "annihilate"

(large, resistive dissipation)

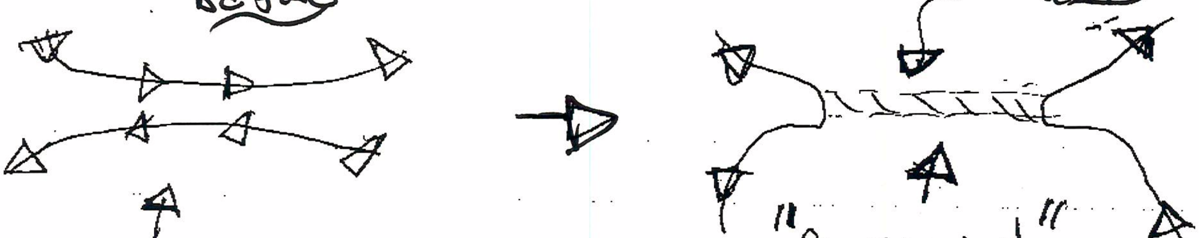
N.B. { A particular V value is required for stationarity

layer

N.B. $\rightarrow$ why "reconnection"?

before

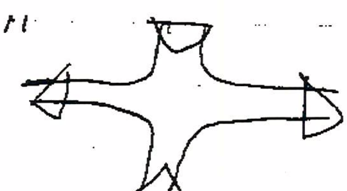
after



initial magnetic configuration

"reconnected" configuration

$\rightarrow$ flow is "stagnation point"



ala' shock

→ How Calculate? → Match In-Flow → Out-Flow

(S_0-P_0 is a great Back-of-Envelope ...)

Conserved: ① – mass ($\underline{\nabla} \cdot \underline{V} = 0$)

② – momentum in $\hat{x}$ direction (symmetry)

③ – energy balance, →

→ rate of field 'delivery' to reconnection region

MUST BALANCE

→ rate of Ohmic dissipation $E \cdot J \sim \eta J^2$

① $\rho_0 \underset{\text{inflow}}{V} \underset{\text{extent in } \hat{x}}{L} = \rho_0 \underset{\text{outflow}}{V_0} \underset{\text{extent in } \hat{y}}{\Delta}$

Mass balance

$\nabla \cdot \underline{V} = 0$

$$VL = V_0 \Delta$$

$$V = V_{out} \Delta / L$$

② $\rho_0 \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right) = -\nabla \left(P + \frac{B^2}{8\pi} \right) + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi}$

$$\underline{V} \cdot \underline{\nabla}\, \underline{V} = -\underline{\nabla} \left(\frac{V^2}{2} \right) + \underline{V} \times \underline{\omega}$$

symmetry: $0 = \underline{\nabla} \left(P + \frac{B^2}{8\pi} + \frac{\rho_0 V^2}{2} \right)$

modified Bernoulli Eqn.

$\underline{a} \rightarrow V = 0$, $\underline{B \text{ finite}}$

$\underline{b} \rightarrow V = V_{out}$, $\underline{B \rightarrow 0}$

$\left(\frac{B^2}{8\pi} \ll P \right)$

So $\quad p + \frac{B^2}{8\pi} + \frac{\rho_0 V^2}{2} = \text{const.}$

$$p + \frac{B^2}{8\pi} = p + \frac{\rho_0 V_{out}^2}{2}$$

$$V_{out}^2 = B^2/4\pi\rho_0 = V_A^2$$

$\hookrightarrow$ Alfven speed

$$\left[\begin{array}{l} V_{out} = V_A \\ V = V_A \dfrac{\Delta}{L} \end{array}\right.$$

specifies "speed V", in terms Δ.

③ 

Energy balance

$\Rightarrow$

$$\rightarrow \left[\begin{pmatrix} \text{Rate of Magnetic} \\ \text{Energy Inflow} \end{pmatrix} = \begin{pmatrix} \text{Rate of Ohmic} \\ \text{Dissipation, net} \end{pmatrix} \right]$$

$P_{OH} = \frac{J^2}{\sigma}\Delta L$ so $\dot{E}_{OH} = \frac{J^2}{\sigma} L\Delta$

$$= \left(\frac{c}{2\pi}\right)^2 \frac{B^2}{\Delta^2}\frac{L\Delta}{\sigma}$$

$\nabla \times B = \frac{4\pi}{c} J$

$2B = \frac{4\pi}{c} J \Delta$

$$P_{in} = 2\left(B^2/8\pi\right)VL = \dot{E}_{in}$$

balance $\Rightarrow \cancel{2}\left(\frac{\cancel{B^2}}{\cancel{8\pi}}\right) V \cancel{L} = \frac{c^2}{\cancel{4\pi}\,4\pi\sigma}\frac{\cancel{B^2}}{\Delta^2}\cancel{L}\Delta$

$$V = \left(\frac{c^2}{4\pi\sigma}\right)/\Delta \sim \frac{\eta}{\Delta}$$

$$\frac{c^2}{4\pi\sigma} \equiv \eta \left(\sim \frac{L^2}{T}\right)$$

$V = V_A \cdot \Delta/L$

$V = \eta/\Delta$

$\Rightarrow \quad \frac{\Delta}{L} = \left(\frac{\eta L}{V_A}\right)^{1/2} = \left(1/R_m\right)^{1/2}$

$R_m = \frac{VL}{\eta} \equiv$ Magnetic Reynolds # (here with $V = V_A$)

and $\boxed{V = V_A/\sqrt{R_m}}$

$\Rightarrow$ Punch Line : (for large R_m)

1. – layer is thin $\quad \frac{\Delta}{L} \sim 1/\sqrt{R_m}$

   – speed is faster than η/L, slower than V_A $\quad \underline{V \sim V_A/\sqrt{R_m}}$

2. Flow pattern is a la' stagnation $\Rightarrow$ ejection from reconnection layer at V_A

Moral of this story: $\quad \frac{1}{\tau_R} \sim \frac{V}{L} \sim \frac{(V_A)^{1/2} \eta^{1/2}}{L^{3/2}}$

→ freezing-in violated when flows bring opposing $\underline{B}$ into contact $\overrightarrow{\quad} \; \overleftarrow{\quad}$.

$\Rightarrow$

→ generates singularities → thin current layers, which alter initial magnetic topology.

$\Rightarrow$ "magnetic reconnection", "tearing", etc.

→ Magnetic Helicity

- another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x \; \underline{A} \cdot \underline{B}$$

V is taken to be the volume of a 'flux tube'.

- what, yet another invariant!?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x \; A \cdot \underline{\nabla} \times \underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

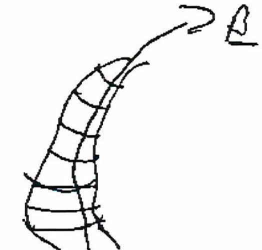
→ K is a pseudo-scalar
∴ has orientation or "handedness"...

Proceed via:
- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x \, \underline{\nabla}\chi \cdot \underline{B}$$



$$= K + \int_V d^3x \, \underline{\nabla} \cdot (\underline{B}\chi)$$

$= 0$, to surface term. $\left\{ \underline{B} \cdot \hat{n} = 0 \right.$ on surface of tube

Now, consider a blob of MHD fluid in motion

can show $\dfrac{dK}{dt} =$

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{V} \times \underline{\nabla} \times \underline{A} - c\nabla\phi - c\eta \underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{V} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V} + \eta \nabla^2 \underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt} \int_V d^3x \, (\underline{A} \cdot \underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt} \cdot \underline{B} + \underline{A} \cdot \frac{d\underline{B}}{dt} \right) + \int \underline{A} \cdot \underline{B} \, \frac{d}{dt} d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + (\underline{V}\cdot\underline{\nabla}\,\underline{A})\cdot\underline{B} + \underline{A}\cdot\frac{\partial \underline{B}}{\partial t} + \underline{A}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{B})\right) + \underline{A}\cdot\underline{B}\;\underline{\nabla}\cdot\underline{V} \Big)$$

where $\frac{d}{dt} d^3x = \underline{\nabla}\cdot\underline{V}$

i.e. $\frac{d}{dt} dV = \frac{d}{dt} d\underline{\sigma}\cdot d\underline{\ell} + d\sigma \cdot \frac{d}{dt} d\underline{\ell}$

$$= -d\underline{\ell}\cdot\underline{\nabla}\underline{V}\cdot d\underline{\sigma} + (\underline{\nabla}\cdot\underline{V})(d\underline{\sigma}\cdot d\underline{\ell}) + d\underline{\ell}\cdot\underline{\nabla}\underline{V}\cdot d\underline{\sigma}$$

$$= \underline{\nabla}\cdot\underline{V}\; d^3\cancel{x}$$

s.t. and $\underline{B}\cdot\hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \Big[\big(\underline{B}\cdot\cancel{\underline{V}}\times\underline{B} - c\,\cancel{\underline{B}}\cdot\underline{\nabla}\phi - c\eta\,\underline{J}\cdot\underline{B}\big) + \underline{A}\cdot\big(\underline{\nabla}\times(\underline{V}\times\underline{B})\big) + \underline{\nabla}\cdot\big((\underline{A}\cdot\underline{B})\,\underline{V}\big) + \underline{A}\cdot\eta\nabla^2\underline{B}\Big]$$

where $\underline{A}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{B}) + \underline{B}\cdot(\underline{V}\cdot\underline{\nabla}\underline{A}) + \underline{A}\cdot\underline{B}\;\nabla\cdot\underline{V} = \nabla\cdot(\underline{V}\,\underline{A}\cdot\underline{B})$

$$\frac{dK}{dt} = \int d^3x \Big[\nabla\cdot\big((\underline{A}\cdot\underline{B})\underline{V}\big) + \nabla\cdot\big((\underline{V}\times\underline{B})\times\underline{A}\big) + (\underline{V}\times\cancel{\underline{B}})\cdot(\underline{\nabla}\times\underline{A})$$

$$- c\eta\,\underline{J}\cdot\underline{B} - \eta(\underline{A}\cdot\underline{\nabla}\times\underline{J})\,c \Big]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla} \cdot \left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta(\underline{A}\times\underline{J}) \right] - c\eta\,\underline{J}\cdot\underline{B} - c\eta\,\underline{J}\cdot\underline{B} \right\}$$

$$= \int d\underline{S}\cdot\left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta\,\underline{A}\times\underline{J} \right]$$

$$- 2\int d^3x \left[c\eta\,\underline{J}\cdot\underline{B} \right]$$

$$= \int d\underline{S}\cdot\left[\cancel{(\underline{A}\cdot\underline{B})\underline{v}} - \cancel{(\underline{A}\cdot\underline{B})\underline{v}} + \underbrace{(\underline{A}\cdot\underline{v})\,\underline{B}}_{0} \right] - c\eta\int d\underline{S}\cdot\underline{J}\times\underline{A}$$

$\underline{B}\cdot\underline{n} = 0$, on tube

$$- 2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$$= -\int c\eta\, d\underline{S}\cdot\left[\underline{\nabla}\underline{B}\cdot\underline{A} - \cancel{\underline{A}\cdot\underline{\nabla}\,\underline{B}} \right] - 2c\eta\int d^3x\,\underline{J}\cdot\underline{B}$$

$$= -2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta\int d^3x\,(\underline{J}\cdot\underline{B})}$$

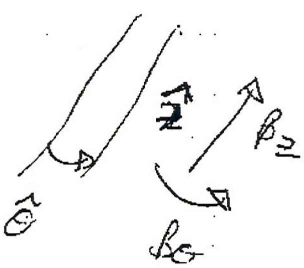
and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular $\underline{J}$)

$\therefore$ Magnetic Helicity is conserved in ideal MHD.

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

interesting to note: $q(r) = \frac{r B_z}{R B_\theta(r)} = \frac{1}{R \mu(r)}$



$\mu(r) = \frac{B_\theta(r)}{r B_z}$ → field line pitch.

(length scale over which winding varies)

cylindrical plasma → $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r} \int_0^r r B_z \, dr$

$A_z = - \int_0^r B_\theta \, dr$

so $$\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z \, dr - B_z \int_0^r B_\theta \, dr$$

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu} \, dr - B_z \int_0^r B_\theta \, dr$$

$$\underline{A}\cdot\underline{B} = B_z \left[\mu \int_0^r \frac{B_\theta}{\mu} \, dr - \int_0^r B_\theta \, dr \right]$$

$= 0$ for constant μ

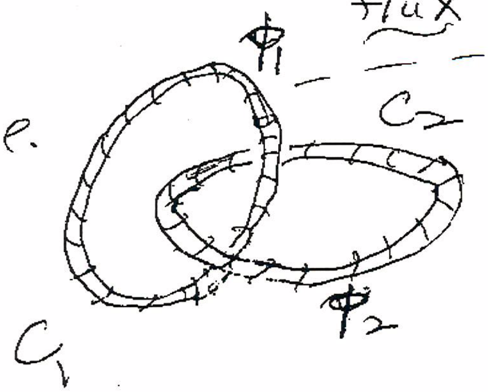
$\therefore$ non-zero helicity requires $\mu = \mu(r)$

i.e. — pitch varies with radius

$\Rightarrow$ $\boxed{\text{magnetic shear}}$ / twist

— physically $\rightarrow$ helicity means self-linkage of 2 flux tubes

i.e.



tube 1: flux $\Phi = \int \underbrace{d\underline{A}}_{\text{x-section area}} \cdot \underline{B} = \underbrace{\phi_1}_{\text{const}}$

tube 2: $\Phi = \phi_2$

field in loops, only

Now, for volume V_1 of tube 1

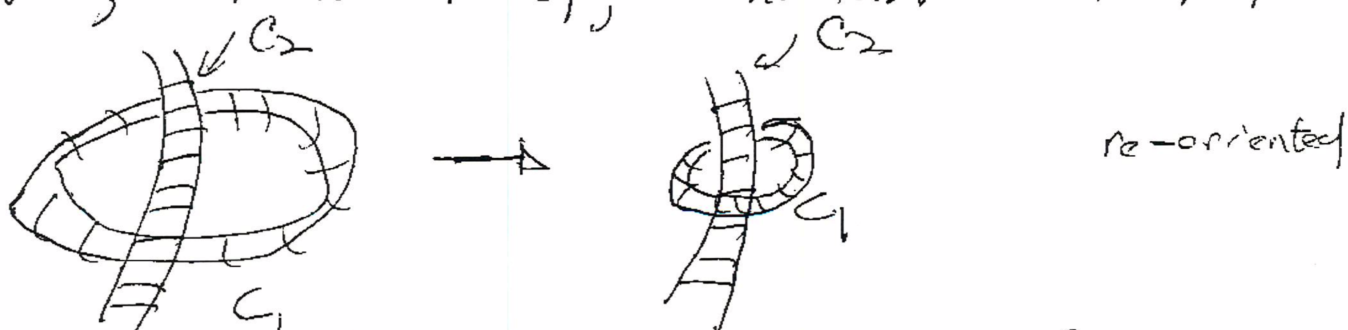
$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \oint_{C_1} d\ell \int_{A_1} dS\ A\cdot\underline{B}$$

C_1 — along loop; A_1 — X-sect area

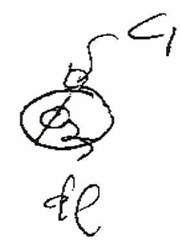
$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\, d\cancel{A}$$

$$= \Phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



$\Rightarrow$ in X section:



but $\int_{C_1} A\cdot d\ell = \int_{A\ \text{enclosed}} B\cdot dS = \Phi_2$

so . . . $K_1 = \Phi_1 \Phi_2$ $\Rightarrow$ product of fluxes

similarly $K_2 = \Phi_2 \Phi_1$

$\therefore$ $K = 2\Phi_1 \Phi_2$

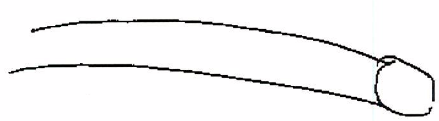
if n windings $K = K_1 + K_2 = \pm 2n\Phi_1 \Phi_2$.

$\Rightarrow$ helicity is measure of self-linkage of magnetic configuration.

Why care $\rightarrow$ Taylor Conjecture (1974) (J.B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP



~ toroid
~ toroidal current

well fit by
$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$
$\Leftarrow$
force free

$\Rightarrow$ why so robust?, especially since RFP so turbulent

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point
- helicity conserved in flux tubes, to η
- toroidal plasma $\Rightarrow$ many small tubes

etc.

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but global helicity conserved.

i.e.

$$\underbrace{\int}_{\text{plasma volume}} \underline{A}\cdot\underline{B}\, d^3x = K_0 \rightarrow \text{conserved.}$$

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\,\frac{B^2}{8\pi} + \lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right] = 0$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity.

Comments:

→ it works! - indeed amazingly well - for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof.

Hypothesis: Selective Decay

- energy cascades → small scale
- helicity cascades → large scale (less dissipation)

- Relevance to driven system?
i.e. in real RFP, transformer on.

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing..