Mean Field Electrodynamics – A Brief Introduction

→ discussion of relaxation ⇒

$$-\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \frac{\langle \underline{B} \rangle}{\langle B \rangle^2} \, \underline{\nabla} \cdot \vec{\Gamma}_H \, ; \quad \vec{\Gamma}_H = -\lambda D \left(\langle J_{\parallel} \rangle / \langle B \rangle \right)$$

for consistency with Taylor hypothesis.

⇒ but, how calculate $\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle$ – i.e. what form does mean field EMF actually have?

→ problem in mean field electrodynamics (i.e. closure, akin Q.L.T.).

→ some simple cases:

- fluid turbulence + weak $\langle \underline{B}_0 \rangle$
- $R_m < 1$.

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \langle \underline{v} \times \underline{\tilde{B}}^{(1)} \rangle$$

response of $\tilde{\underline{B}}$ to $\tilde{\underline{v}}$, in presence $\langle \underline{B}_0 \rangle$

then

$$\partial_t \tilde{B} + \overbrace{\nabla\times\tilde{v}\times\tilde{B}}^{NL\,\to\,\Delta\omega_k} - \overbrace{\eta\nabla^2\tilde{B}}^{\text{resistive diffn, } \eta k^2}$$

$$= \langle B\rangle\cdot\nabla\tilde{v} - \tilde{v}\cdot\nabla\langle B\rangle$$

$$\therefore \quad (-i\omega + \eta k^2)\,\tilde{B}_{k,\omega} = \underbrace{i\underline{k}\cdot\langle B\rangle\,\tilde{v}_{k,\omega}}_{\text{bending}} - \underbrace{\tilde{v}_{k,\omega}\cdot\nabla\langle B\rangle}_{\text{field advection}}$$

$$\tilde{B}_{k,\omega} = \frac{i\underline{k}\cdot\langle B\rangle\,\tilde{v}_{k,\omega} - \tilde{v}_{k,\omega}\cdot\nabla\langle B\rangle}{-i\omega + \eta k^2}$$

so

$$\langle\tilde{v}\times\tilde{B}\rangle = \sum_{k,\omega}\tilde{v}_{-k,-\omega}\times\frac{\left[\overset{①}{i\underline{k}\cdot\langle B\rangle\,\tilde{v}_{k,\omega}} - \overset{②}{\tilde{v}_{k,\omega}\cdot\nabla\langle B\rangle}\right]}{-i\omega + \eta k^2}$$

② → even in k

⇒ advection of $\langle B\rangle$

i.e. turbulent resistivity

① → odd in $\underline{k}$ ⟷ bending → kink, symmetry breaking, for contribution → physics?
→ ?

N.B.: In both cases, irreversibility provided by resistive diffusion ⇒ otherwise difficulty

For isotropic velocity spectrum:

$$\langle \tilde{v}_i(\underline{k},\omega)\, \tilde{v}_j^*(\underline{k}',\omega') \rangle = \delta(\underline{k}-\underline{k}')\,\delta(\omega-\omega')\, \Phi_{ij}(\underline{k},\omega)$$

$$\Phi_{i,j}(k,\omega) = \overset{①}{\frac{\underline{E(k,\omega)}}{4\pi k^2}\left(k^2\delta_{ij} - k_i k_j\right)} + \overset{②}{\frac{iF(k,\omega)}{8\pi k^2}\,\epsilon_{ijk}k_k}$$

① → energy density, even power
→ $\underline{\nabla}\cdot\underline{v} = 0$

② Now,

$$F(k,\omega) = i\int \epsilon_{ike}\, k_k\, \Phi_{i,e}(k,\omega)\, dS_{\underline{k}}$$

so

$$\langle \underline{v}\cdot\underline{\omega}\rangle = i\epsilon_{jkl}\iint k_k \Phi_{jl}(k,\omega)\, d\underline{k}\, d\omega$$

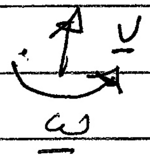
$\downarrow$

spectral helicity $= \iint dk\, d\omega\, F(k,\omega)$

$\Rightarrow$

(2) $\sim \langle \underline{v}\cdot\underline{\omega}\rangle$

$\Rightarrow$ turbulence helicity (mean projection $\underline{v}$ on $\underline{\omega}$)



so, after some crank (see Moffat: available free, online):

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}}\rangle = \alpha\langle \underline{B}\rangle - \beta\langle \underline{J}\rangle$$

$$\alpha \equiv -\frac{1}{3}\eta \iint \frac{dk\, d\omega\, k^2 F(k,\omega)}{\omega^2 + (\eta k^2)^2}$$

$\Rightarrow$ α is weighted integral of helicity spectrum

i.e.

$\sim \langle \underline{v}\cdot\underline{\omega}\rangle$

$$\beta = \frac{2}{3}\eta \iint dk\, d\omega \, \frac{k^2 E(k,\omega)}{\omega^2 + (\eta k^2)^2}$$

$\Rightarrow$ β is weighted integral of energy spectrum

i.e.

$\sim \langle \tilde{v}^2 \rangle$

so

$$\frac{\partial \langle \underline{B} \rangle}{\partial t} - \eta \nabla^2 \langle \underline{B} \rangle = \underline{\nabla} \times \underbrace{\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle}_{\text{mean EMF}}$$

$$\underbrace{\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle}_{\text{mean EMF}} = \underbrace{\alpha \langle \underline{B} \rangle}_{\alpha\text{-effect}} - \underbrace{\beta \langle \underline{J} \rangle}_{\beta\text{-effect}}$$

$\therefore$

$$\frac{\partial \langle \underline{B} \rangle}{\partial t} - \eta \nabla^2 \langle \underline{B} \rangle = \alpha \, \underline{\nabla} \times \langle \underline{B} \rangle + \beta \nabla^2 \langle \underline{B} \rangle$$

$\therefore$

$\Rightarrow$ β as turbulent resistivity $\Rightarrow$ random advective mixing of $\langle B \rangle$.

$\Rightarrow \alpha$?

Further interesting to note:

$\rightarrow$ look for force-free condition fields:

$$\underline{\nabla} \times \langle \underline{B} \rangle = \lambda \langle \underline{B} \rangle$$

$$\partial_t \langle \underline{B} \rangle - (\eta + \beta) \nabla^2 \langle B \rangle = \alpha \lambda \underline{\langle B \rangle}$$

$\Rightarrow$

$$\gamma_B = \alpha \lambda - (\eta + \beta) \lambda^2$$

$\therefore$ α can amplify field $\rightarrow$ depending on λ (scale)

$\Rightarrow$ $\boxed{\text{dynamo}}$ $\leftrightarrow$ via α - effect!

$\rightarrow$ Physics:

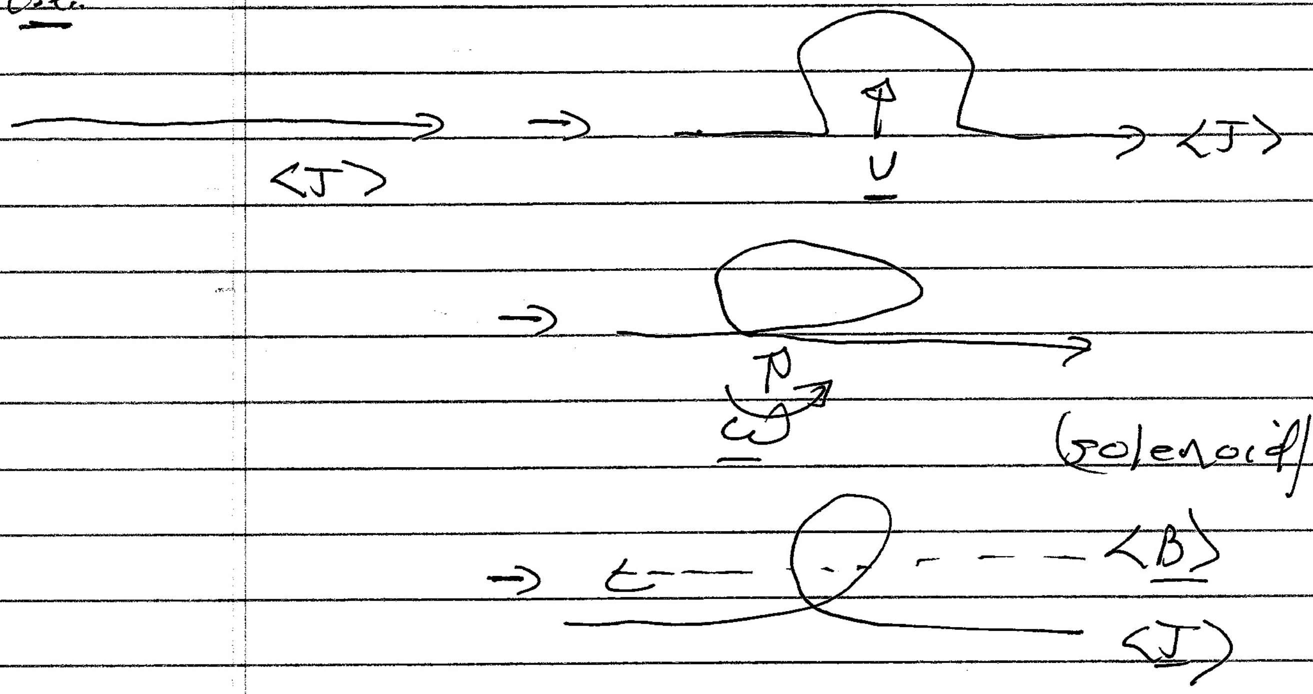
$\alpha \rightarrow$ helicity $\leftrightarrow$ $\langle \underline{\hat{v}} \cdot \underline{\hat{\omega}} \rangle$

Then:

$\rightarrow$ can stetch, twist, fold lines, to amplify field.

→ α : $\langle J \rangle \rightarrow \langle \underline{B} \rangle$

i.e.



$$\langle \underline{B} \rangle \parallel \langle J \rangle .$$

repeat ⇒ amplify field.

i.e. $\gamma_B = \alpha \lambda - (\eta + \beta) \lambda^2$

$$d\gamma / d\lambda = \alpha - 2(\eta+\beta)\lambda = 0$$

$$\lambda_{\substack{max \\ growth}} = \alpha / 2(\eta + \beta)$$

$$\gamma_B\Big|_{max} = \alpha^2 / 4(\eta + \beta)$$

$\nearrow$

$\sim \alpha^2$ dynamo.

N.B:

→ locking-in (reconnection!) crucial!
  ⇒ role of η as cross-phase.
  ⇒ high Rm → problematic.

→ non-linearity, especially high Rm
  ⇒ a field in itself!

N.B.: Impact/role of magnetic helicity
  in dynamo control.