Chapter 2

An Introduction to Mean Field Dynamo Theory

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2.1   Introduction

Magnetic fields are essentially ubiquitous, being detected over a tremendous range of scales in planets, stars, accretion discs and in the interstellar medium. The dynamic behaviour of such fields is then responsible for a vast range of astrophysical phenomena (see, for example, Parker 1979). For instance, the solar magnetic field gives rise to sunspots, solar flares and coronal mass ejections; it also plays a major role in shaping the solar wind which, on interacting with the Earth's magnetic field, causes aurorae. Starspots, analogous to sunspots but covering a much greater surface area, have been detected on a number of cool stars. The pulsed emission of pulsars is a consequence of an extremely strong magnetic field. On the largest scales, the interstellar magnetic field plays a role in star formation, mediating angular momentum transport as the star collapses. The ultimate question in the study of astrophysical magnetic fields must then be that of the *origin* of the magnetic field in cosmical objects. In particular, one might ask whether the observed magnetic fields are simply 'fossil fields', or whether, alternatively, they are being continually regenerated — i.e. whether some sort of *dynamo* process is taking place.

For collision-dominated plasmas with short mean free paths, such as those found in stellar interiors, the evolution of magnetic fields is very well described by the equations of single-fluid magnetohydrodynamics (MHD).

(Although, for example, in studies of stellar atmospheres — such as the solar corona — the use of MHD is not based on such solid foundations.) In this review we shall be concerned only with physical systems for which the MHD description is appropriate.

The key equation of MHD is the magnetic induction equation, derived from the pre-Maxwell equations (in which the displacement current is neglected) and the simplest form of Ohm's law in a moving conductor, namely

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}), \tag{2.1}$$

where $\mathbf{J}$ is the electric current, $\mathbf{E}$ the electric field, $\mathbf{U}$ the velocity field, $\mathbf{B}$ the magnetic field, and σ the electrical conductivity. The induction equation takes the form:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{2.2}$$

where η $(= 1/\mu_0\sigma$, where μ_0 is the magnetic permeability) is the magnetic diffusivity (assumed constant in this derivation). The first term on the right hand side of Eq. (2.2) describes the advection of the magnetic field by the velocity, the second term describes collisional dissipation of the field. An order of magnitude estimate of the relative size of these two terms is given by the magnetic Reynolds number $Rm = UL/\eta$, where U and L are a typical velocity and length. Significantly, in most astrophysical contexts Rm is huge.

Clearly, in the absence of any fluid motions the induction equation simply reduces to a vector diffusion equation, with a timescale for decay of the magnetic field (the Ohmic timescale) given by $\tau_\eta \sim L^2/\eta$. For the Earth, taking L as the radius of the metallic core and η as the appropriate molecular value, the Ohmic decay time τ_η is of the order of 10^4 years, several orders of magnitude shorter than the timescale over which the Earth's magnetic field has existed, namely $O(10^9)$ years (see, for example, Backus *et al.* 1996). Thus it follows immediately that the advective term in Eq. (2.2) is of paramount importance — any fossil field would by now have decayed to a negligible strength. In other words, the fluid motions are critical to maintenance of the magnetic field. This is what is meant by a (hydromagnetic) dynamo.

For the case of the Sun, whose magnetic field can be studied in far greater detail than that of the Earth, the Ohmic time is very long (roughly speaking, L is large and η small), and is comparable with the lifetime of the Sun. Solely on these grounds it is not therefore possible to assert that the Sun's field cannot be primordial in origin. However, conversely, it is the

fact that the observed solar field varies on a *short* timescale, with reversals roughly every eleven years, that strongly suggests that the magnetic field is a product of dynamo action. The alternative — an observed rapidly reversing field fed somehow by a (hidden) slowly decaying and non-reversing field (a hydromagnetic *oscillator*) — is much harder to envisage. Magnetic fields with short-term variability have also been observed on a number of other solar-type stars, and it is natural to assume that they are also the result of some kind of dynamo action (see, for example, the review by Rosner 2000).

Within accretion discs, although there is no compelling observational evidence of dynamo action, numerical simulations of the nonlinear development of the magneto-rotational instability, which is believed to play a crucial role in angular momentum transport, suggest a novel 'bootstrapping' dynamo mechanism in which the instability of a weak field leads to turbulent motions, which can then act to amplify the magnetic field (see the review by Balbus & Hawley 1998). On the largest, galactic, scale the question of whether the magnetic field is a product of dynamo action is rather more open (see Zweibel 2005). The Ohmic decay time of the field is about ten orders of magnitude larger than the age of the Universe, so on these grounds the field could obviously be primordial. Although there are possible difficulties in reconciling a wound-up primordial field with the observed spatial scale between galactic field reversals, there is currently no overwhelming observational evidence to decide the issue either way.

A complete mathematical description of MHD dynamo action requires the solution of the induction equation (2.2) and also of the momentum equation, which accounts for the back-reaction of the magnetic field on the fluid flow through the Lorentz force $\mathbf{J} \times \mathbf{B}$. For compressible fluids an energy equation and an equation of state must also be solved. Thus the full problem requires the solution of a set of coupled nonlinear partial differential equations, which represents a formidable mathematical and computational challenge. A simpler, but still non-trivial problem, is to consider the evolution of the magnetic field under a *prescribed* velocity field, which requires solution only of Eq. (2.2). Investigating the growth of the magnetic field governed by this linear equation is known as the *kinematic* dynamo problem; a flow for which the magnetic field grows exponentially (governed solely by Eq. (2.2)) is said to act as a kinematic dynamo.

Determination of kinematic dynamo action, even though it requires solution of only one linear partial differential equation, is nonetheless not straightforward. Indeed, from the initial formulation of the problem in a

short paper by Larmor (1919) (reproduced in Ruzmaikin *et al.* 1988), it
took a period of nearly forty years before the first rigorous demonstrations
of kinematic dynamo action (Herzenberg 1958; Backus 1958). The main
obstacle to success came from the fact that simple fields and flows of the
kind that allow analytical progress with Eq. (2.2) typically do not act as dy-
namos; indeed, there are a number of anti-dynamo theorems that expressly
rule out flows and fields with certain symmetries. The most famous are
due to Cowling (1934), who showed that a purely axisymmetric magnetic
field cannot be maintained by dynamo action, and to Zeldovich (1957), who
showed that dynamo action cannot result from plane two-dimensional mo-
tion. Subsequently there have been many extensions to these fundamental
results (see, for example, Hide & Palmer 1982; Ivers & James 1984). Conse-
quently, the dynamo problem is inherently three-dimensional and, as such,
decidedly non-trivial.

A distinction of a different kind may be drawn between *small-scale* and
large-scale dynamos. A small-scale dynamo is one in which a growing mag-
netic field occurs on scales comparable with or smaller than those of the
driving flow. A large-scale dynamo, on the other hand, has significant en-
ergy on scales very large compared with those of the velocity field. (The
magnetic field of the Sun, which gives rise to sunspots and is global in scale,
is envisaged to be the product of a large-scale dynamo.) However, this dis-
tinction, although sometimes useful, is not clear-cut. A large-scale dynamo
will, typically, also have strong small-scale fields; conversely a small-scale
dynamo flow, *given the opportunity*, will typically generate a certain amount
of large-scale field — i.e. the spectrum of the field will not have an abrupt
cut-off at the driving scale, but will decrease gradually to small wavenum-
bers (large scales). Thus a certain amount of caution is needed in categoris-
ing any dynamo as being of small or large scale. Moreover, in a turbulent
flow there may be a wide range of energetic scales of motion; in such cases
even the idea of a separation of scales becomes unclear.

A major breakthrough in the study of dynamo action occurred with the
formulation of *mean field electrodynamics* by Steenbeck, Krause and Rädler
(1966 and subsequent papers; see the monographs by Krause & Rädler 1980
and Moffatt 1978). This theory considers the generation by dynamo action
of magnetic fields on scales large compared with those of the driving flow,
and is formulated in terms of the large-scale (or mean) component of the
field, with the influence of small-scale motions and field encapsulated in a
few fundamental coefficients of the problem (such as the famous 'α-effect').
In a few special cases these can be calculated exactly; more generally though

they are parameterised in a physically plausible manner. For the past forty years astrophysical dynamo theory has been dominated by mean field electrodynamics.

In this review we aim to provide a brief introduction to mean field electrodynamics. We shall concentrate on the generic aspects of the theory, rather than on object-specific models aimed at describing the magnetic fields of particular astrophysical bodies. The literature relating to mean field electrodynamics is vast, and so we make no claim here to be exhaustive in our choice of references; a fuller reference list may be obtained from the reviews of Roberts & Soward (1992), Ossendrijver (2003) and Brandenburg & Subramanian (2005). In Sec. 2.2 we discuss kinematic considerations — the formulation of mean field electrodynamics, the calculation of the governing coefficients in the theory, and simple linear mean field dynamo models. The incorporation of nonlinear effects is described in Sec. 2.3. Research into mean field electrodynamics remains an active, and indeed controversial, topic in astrophysical MHD; some of the difficulties of the theory and the issues that arise are discussed in Sec. 2.4.

2.2 Kinematic Mean Field Theory

2.2.1 *Formulation*

Mean field electrodynamics is, at heart, a linear (or kinematic) theory, being based on the induction equation. In this section we shall concentrate solely on the linear formulation; nonlinear effects, which may be incorporated through a variety of approaches, are described in Sec. 2.3.

The fundamental assumption underlying mean field electrodynamics is that there exists a strict separation of length scales for both the magnetic and velocity fields. The velocity and magnetic fields have a small-scale component, with characteristic length scale ℓ, and a large-scale — or mean — component with scale $L \gg \ell$. For the case of a turbulent velocity field, ℓ may be regarded as a typical size of the energetic eddies. Averages, which we denote by $\langle \cdot \rangle$, may then be taken over an intermediate length scale a satisfying $\ell \ll a \ll L$. The velocity and magnetic fields may be decomposed into their mean and fluctuating components:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \qquad \mathbf{U} = \mathbf{U}_0 + \mathbf{u}, \tag{2.3}$$

where, by definition, $\langle \mathbf{b} \rangle = \langle \mathbf{u} \rangle = 0$. Averaging the induction equation

(2.2) yields an equation for the evolution of the mean magnetic field:

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{U}_0 \times \mathbf{B}_0) + \nabla \times \boldsymbol{\mathcal{E}} + \eta \nabla^2 \mathbf{B}_0, \tag{2.4}$$

where $\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle$ is the mean electromotive force (emf), resulting from interactions between the small-scale velocity field and the small-scale magnetic field. It is this term that provides the crucial distinction between the (un-averaged) induction equation and the *mean* induction equation. Clearly, in order to make progress with Eq. (2.4) it is necessary to close the system by expressing $\boldsymbol{\mathcal{E}}$ in terms of the mean field $\mathbf{B}_0$ and its derivatives. This is done by considering the equation governing the fluctuating magnetic field, obtained by subtracting Eq. (2.4) from Eq. (2.2):

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{U}_0 \times \mathbf{b}) + \nabla \times (\mathbf{u} \times \mathbf{B}_0) + \nabla \times \mathbf{G} + \eta \nabla^2 \mathbf{b}, \tag{2.5}$$

where $\mathbf{G} = \mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle$. Symbolically, Eq. (2.5) may be written as

$$\mathcal{L}(\mathbf{b}) = \nabla \times (\mathbf{u} \times \mathbf{B}_0), \tag{2.6}$$

where $\mathcal{L}$ is a linear operator. There are two distinct possible modes of behaviour for the small-scale magnetic field $\mathbf{b}$. One is if $\mathbf{b}$ is 'slaved' to the mean field $\mathbf{B}_0$; here $\mathbf{b}$ is driven solely by the inhomogeneous term $\nabla \times (\mathbf{u} \times \mathbf{B}_0)$ — in the absence of $\mathbf{B}_0$ the small-scale field simply decays to zero. The second is when the operator $\mathcal{L}$ can support non-trivial solutions even in the absence of a large-scale field $\mathbf{B}_0$ — in other words, the velocity field supports small-scale dynamo action. As we shall see, the implications for mean field theory for the two cases can be rather different.

Let us consider first the case when $\mathbf{b}$ exists only if the right hand side of Eq. (2.6) is non-zero; it is then linearly and homogeneously related to $\mathbf{B}_0$. Hence, for a given velocity field $\mathbf{u}$, the mean emf $\boldsymbol{\mathcal{E}}$ is also linearly and homogeneously related to $\mathbf{B}_0$. Since, by construction, the mean magnetic field varies on a long length scale, one may therefore postulate an expansion for $\boldsymbol{\mathcal{E}}$ of the form

$$\mathcal{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \cdots . \tag{2.7}$$

For consistency with Eq. (2.4), which is a second order partial differential equation, it is important that the first two terms (and only the first two terms) of this expansion are retained. Since the vector $\boldsymbol{\mathcal{E}}$ is a polar vector, whereas $\mathbf{B}_0$ is an axial vector, it follows therefore that α_{ij} and β_{ijk} are *pseudo*-tensors (i.e. they change their sign, compared with tensors, under transformation). Their physical interpretation is brought out most clearly

by considering the case of isotropic turbulence. In the kinematic regime the pseudo-tensors α_{ij} and β_{ijk} can depend only on the statistical properties of the flow field and on the diffusivity η; therefore, for isotropic turbulence, they must take the form

$$\alpha_{ij} = \alpha\delta_{ij}, \quad \beta_{ijk} = \beta\epsilon_{ijk}, \tag{2.8}$$

where α is a pseudo-scalar and β a true scalar. Substitution from Eq. (2.8) into Eq. (2.4) gives

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{U}_0 \times \mathbf{B}_0) + \nabla \times \alpha\mathbf{B}_0 + (\eta + \beta)\nabla^2\mathbf{B}_0, \tag{2.9}$$

where we have made the further simplifying assumption that β is constant. Clearly β is an additional, turbulent, contribution to the magnetic diffusivity. On physical grounds we expect this to be positive for a genuinely turbulent flow; however it is of interest to note that there are certain types of flow (e.g. random potential flows) for which β is negative (Krause & Rädler 1980).

The most important term though is $\nabla \times \alpha\mathbf{B}_0$, the '$\alpha$-effect' of mean field dynamo theory, which may lead to growth of the mean magnetic field. Whereas in the full induction equation (2.2) the advective term has the form of the curl of a vector *perpendicular* to the magnetic field, in the mean induction equation (2.9) the new (α) term is the curl of a vector *parallel* to the mean field. This turns out to have profound consequences. One is that Cowling's theorem no longer holds (because it *is* possible to have dynamos that are axisymmetric *on average*); the problem then becomes much simpler, analytically and computationally, and allows ready progress to be made in dynamo modelling. We explore some such models in Sec. 2.2.3. However, even without any calculation, it is possible to gain considerable insight into the nature of any turbulence that can lead to a non-zero value of α. Consider the case when the turbulence is *reflectionally symmetric*; i.e. its statistics have no sense of 'handedness'. Then, on changing from a right-handed to a left-handed frame of reference, the quantity α, which depends only on the statistics of the turbulence, is invariant. Conversely though, α, being a pseudo-scalar, must change sign under this reflection of coordinates. Thus it follows that α must vanish for reflectionally symmetric turbulence. Equivalently, α can be non-zero only for turbulence possessing some sense of handedness. In nature, reflectional symmetry is broken in any rotating system. The simplest measure of the lack of reflectional symmetry in a flow is the *helicity*, $H = \langle \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle$.

For non-isotropic turbulence it is instructive to split the α_{ij} tensor into its symmetric and antisymmetric parts. The symmetric pseudo-tensor $\alpha_{ij}^{(s)}$ may be referred to its principal axes:

$$\alpha_{ij}^{(s)} = \begin{pmatrix} \alpha^{(1)} & & \\ & \alpha^{(2)} & \\ & & \alpha^{(3)} \end{pmatrix}. \qquad (2.10)$$

Arguing as above, $\alpha^{(1)}$, $\alpha^{(2)}$ and $\alpha^{(3)}$ must vanish for turbulence that lacks reflectional symmetry.

The antisymmetric part of α_{ij} may be written as $\alpha_{ij}^{(a)} = -\epsilon_{ijk}\gamma_k$, leading to a contribution $\boldsymbol{\gamma} \times \mathbf{B}_0$ to the emf. From comparison with Eq. (2.4), it can be seen that $\boldsymbol{\gamma}$ is to be regarded as a velocity acting on the mean magnetic field; it can be non-zero provided the statistics of $\mathbf{u}$ are either anisotropic or inhomogeneous, but is not dependent on the symmetry properties of the turbulence (Cattaneo *et al.* 1988). A detailed review of the transport effects of turbulence on both scalar and vector fields is given by Moffatt (1983).

For any given flow, one can calculate α_{ij} (at least in principle, and often in practice) by imposing a uniform field and measuring the resulting emf; here the scale separation is between the scale of variation of the flow field and that of the uniform magnetic field (which is formally infinite). This is not a dynamo calculation as such, since the uniform field remains constant in time. However, the resulting value of α_{ij} can then be used (once β_{ijk} is known) to calculate the growth of magnetic fields of large but finite scales. Conceptually (though with great difficulty in practice) β_{ijk} may, provided dynamo action is not taking place, be calculated by considering the diffusion of a magnetic field that is non-uniform but has a scale of variation far in excess of the characteristic scale of velocity. If, however, the flow acts as a large-scale dynamo then such a field will of course not decay but will grow.

The discussion above has focused on the case where the fluctuating field $\mathbf{b}$ is driven by the $\nabla \times (\mathbf{u} \times \mathbf{B}_0)$ term in Eq. (2.5). However, turbulent flows, regardless of their symmetry properties, typically act as small-scale dynamos at sufficiently high values of Rm; i.e. $\mathbf{b}$ can grow quite happily of its own accord with $\mathbf{B}_0 = 0$. The growth rate of such a dynamo will then be determined not by α and β but by completely different properties of the flow, such as its Lyapunov exponents (see, for example, the monograph by Childress & Gilbert 1995). The difficulty comes in determining the nature of the dynamo action that may result from a helical, turbulent flow at high Rm — the typical astrophysical situation. If large-scale growth is

forbidden, through restricting the size of the domain and through the choice of boundary conditions, any dynamo action that ensues will be unambiguously small-scale. However, if the same flow is embedded in an extended domain there are two possibilities: a modified version of the small-scale dynamo with some magnetic energy on the large scales, or alternatively a genuine large-scale — but *not* small-scale — dynamo, though this too will have associated with it strong small-scale fields. Thus, given one particular realisation of kinematic dynamo action in an extended domain, drawing a meaningful distinction between these two types of dynamo may not be possible.

2.2.2 *Calculation of the transport coefficients*

Calculation of the mean emf $\mathcal{E}$, and hence of the tensors α_{ij} and β_{ijk}, requires solution of Eq. (2.5) for the fluctuating magnetic field $\mathbf{b}$. Unfortunately, for a general turbulent flow this is impossible. There are, however, certain limiting cases in which $\mathcal{E}$ *can* be calculated — or at least well-approximated. These results can be complemented by numerical solutions — away from the limiting regimes — for particular flows, although it is not possible to probe the astrophysical regime of extremely high Rm.

The term that renders Eq. (2.5) so troublesome is $\nabla \times \mathbf{G}$. It is therefore obviously of interest to consider situations in which this term can safely be ignored. (The neglect of this term is sometimes referred to as the *first order smoothing approximation* or the *second order correlation approximation*.) Suppose we denote the time and length scales of the velocity field by τ and ℓ respectively, and the rms velocity by u. Then if either $Rm = u\ell/\eta \ll 1$ or $S = u\tau/\ell \ll 1$ (where S is the Strouhal number), the $\nabla \times \mathbf{G}$ term can be neglected.

If $Rm \ll 1$ Eq. (2.5) for $\mathbf{b}$ (assuming, for simplicity, $\mathbf{U}_0 = 0$) simplifies to

$$0 = \nabla \times (\mathbf{u} \times \mathbf{B}_0) + \eta \nabla^2 \mathbf{b}, \qquad (2.11)$$

assuming that dissipative effects are so strong that the time-dependence of $\mathbf{b}$ can be neglected. As described by Moffatt (1978), Eq. (2.11) can be solved via Fourier transforms to give (for isotropic turbulence)

$$\alpha = -\frac{1}{3\eta} \int \frac{F(k)}{k^2} dk, \qquad (2.12)$$

where $F(k)$ is the helicity spectrum function. The corresponding expression for β is

$$\beta = \frac{2}{3\eta} \int \frac{E(k)}{k^2} dk, \qquad (2.13)$$

where $E(k)$ is the energy spectrum function. If, on the other hand, $S \ll 1$ then **b** is governed by

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}_0), \tag{2.14}$$

leading to the expressions

$$\alpha = -\frac{\tau}{3} \langle \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle \qquad \text{and} \qquad \beta = \frac{\tau}{3} \langle \mathbf{u}^2 \rangle \tag{2.15}$$

(see Krause & Rädler 1980). The significant feature of expressions (2.15) is that they are independent of the magnetic diffusivity η. Analogous, but more complicated expressions can be obtained for α_{ij} and β_{ijk} for non-isotropic turbulence; these are discussed at length in Krause & Rädler (1980). New physical effects can result, but they fall outside the scope of this introductory review.

The crucial points to note from the above expressions are that α is directly related to the flow helicity (it is proportional to the helicity in Eq. (2.15) and to a weighted average of the helicity in Eq. (2.12)), and that β is determined by the energy of the flow. The clear link between α and the flow helicity can be understood in terms of Parker's picture of helical motions that raise and twist field lines to provide a net current anti-parallel to the initial field (Parker 1955), as sketched in Fig. 2.1. It is clear though that for this picture to be valid the loops must undergo only small twists — which will be true provided that either diffusion dominates ($Rm \ll 1$) or the motions act only for a very short time ($S \ll 1$). Astrophysical turbulence, however, is characterised by $Rm \gg 1$ and $S = O(1)$; it is therefore of fundamental importance to see whether the close link between the helicity of the flow and the α-effect carries through into this regime.

A very different approach can be taken by the formal neglect of the diffusive terms in the induction equation. The magnetic field is then frozen into the (perfectly conducting) fluid and can be expressed in terms of its initial configuration via the Cauchy solution. This leads to the following expression for α in terms of the Lagrangian displacement $\boldsymbol{\xi}$ (Moffatt 1974):

$$\alpha = -\frac{d}{dt} \langle \boldsymbol{\xi} \cdot \nabla \times \boldsymbol{\xi} \rangle. \tag{2.16}$$

It is of interest to note that the helicity again appears, albeit in a Lagrangian sense. However, it is by no means straightforward to calculate α from Eq. (2.16), and indeed it is not clear that the expression converges as $t \to \infty$. One might hope that expression Eq. (2.16), derived under the assumption of infinite Rm, sheds some light on the behaviour for large but
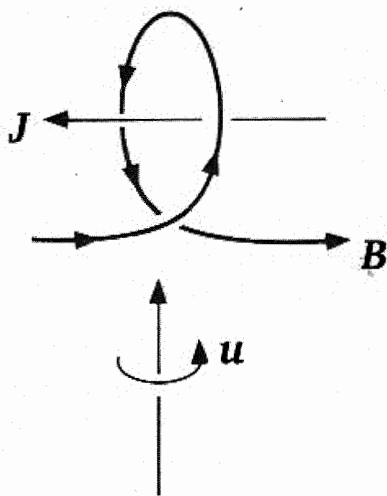
Fig. 2.1 Distortion of the magnetic field by a rising, twisting motion (after Moffatt 1978). For this small value of the twist the associated current is anti-parallel to the initial magnetic field.

finite Rm. If so then this would provide a possible avenue into exploring the astrophysically important case of $Rm \gg 1$. However, the link between the cases of Rm infinite and Rm large but finite is a subtle one that is not fully understood. Expression Eq. (2.16) thus currently remains a relatively unexplored approach to understanding the α-effect.

In order to investigate the relationship between the α-effect and the properties of the velocity field, many calculations of α_{ij} have considered spatially periodic, helical motions. This was an idea pioneered by G.O. Roberts (1972), who considered two-dimensional, incompressible, steady flows, i.e.

$$\mathbf{u}(x,y) = \nabla \times (\psi \hat{\mathbf{z}}) + w \hat{\mathbf{z}}. \tag{2.17}$$

Such flows have the simplifying feature that the induction equation (2.2) supports monochromatic solutions of the form

$$\mathbf{B}(x,y,z,t) = \widetilde{\mathbf{B}}(x,y) e^{pt+ikz}, \tag{2.18}$$

where $\widetilde{\mathbf{B}}$ has the same spatial periodicity as the flow. However they also pose a difficulty in defining precisely what is meant by a large-scale magnetic field. For any given value of z a field of the form Eq. (2.18) has a component that is independent of x and y. Clearly, in some sense, this is a large-scale component — it has no variation in x or y and so has an infinite length

scale in these directions. Its evolution can be described through an α-effect in which the averaging procedure involves averaging over planes of constant z. Conversely though, this purely z-dependent component of the magnetic field may be thought of as simply part of the 'small-scale' eigenfunction $\widetilde{\mathbf{B}}(x, y)$. This indeterminacy seems unavoidable for these special — but computationally tractable — flows.

Roberts (1972) calculated the α-effect for four different flows in the limit as $k \to 0$, exploiting symmetry arguments to determine the form of the anisotropic tensor α_{ij}. He focused particularly on the case of

$$\psi = \sin x \sin y, \quad w = K\psi, \qquad (2.19)$$

using analytical and numerical solutions to obtain α, and hence also the dynamo growth rate, for a range of Rm. The nature of the α-effect for the Roberts flow in the limit of $Rm \to \infty$ was considered by Childress (1979) and later by Soward (1987), who showed that the growth rate scales as

$$p \sim \frac{\ln(\ln Rm)}{\ln Rm}. \qquad (2.20)$$

Childress & Soward (1989) extended their analysis to calculate the α-effect for large Rm for the 'cat's eye' flow with $\psi = \sin x \sin y + \delta \cos x \cos y$. Plunian & Rädler (2002) considered the k-dependence of the α-effect for the Roberts flow, and the possible relationship between this dynamo and that of the Karlsruhe dynamo experiment.

In order for a dynamo to be 'fast' — i.e. to have a positive growth rate in the limit as $Rm \to \infty$ — it is necessary that it has chaotic particle paths (Vishik 1989; Klapper & Young 1995). Steady two-dimensional flows are integrable and hence, as such, cannot act as fast dynamos — although, as for the Roberts flow analysed by Soward (1987), the growth rate may be an extremely slowly decaying function of Rm. It is of interest therefore to consider the α-effect produced by non-integrable flows as $Rm \to \infty$. A much-studied flow in the context of possible fast dynamo action is the time-dependent version of the Roberts flow introduced by Galloway & Proctor (1992), with

$$\psi = \sin(x + \epsilon \cos \omega t) + \cos(y + \epsilon \sin \omega t), \quad w = \widehat{K}\psi. \qquad (2.21)$$

(When $\omega = 0$ flows Eq. (2.19) and Eq. (2.21) are equivalent after a simple transformation.) Courvoisier *et al.* (2006) have investigated the nature of the α-effect for the flow Eq. (2.21), for both finite and zero k, and demonstrated that, for this time-dependent flow, α is a sensitive function of Rm which, furthermore, does not tend to zero for large Rm.

Another widely used approach to investigating the kinematic α-effect, which is one step closer to astrophysical reality, is to consider flows that are driven by rotating thermal convection. Here the flow is not specifically prescribed, but results from the nonlinear evolution of a convective instability; the spatial and temporal dependence of the flow depends on the parameters of the problem (such as the Rayleigh number) and on the boundary conditions. The simplest configuration is the plane layer model, with the rotation axis parallel to gravity, which was investigated in the pioneering studies of Childress & Soward (1972) and Soward (1974), who explored the nature of dynamo action for mildly supercritical Boussinesq convection under the assumptions of first order smoothing. The antisymmetry of Boussinesq convection about the mid-plane leads to an antisymmetric helicity distribution and hence an antisymmetric α— where the large-scale field is horizontal and dependent solely on the vertical coordinate. Weakly supercritical convection is characterised by a relatively simple planform. Soward (1974) was able to demonstrate large-scale dynamo action for some simple steady flows — though the very simplest planform consisting of a single roll with a single horizontal wavenumber does not exhibit dynamo action. The plane layer dynamo has been investigated further numerically (kinematically and dynamically), particularly in the regime of rapid rotation, by Jones & Roberts (2000), Rotvig & Jones (2002) and Stellmach & Hansen (2004). The convection in all of these studies, although outside the weakly nonlinear regime considered by Soward, is nonetheless fairly well-ordered; it gives rise to a significant α-effect and, hence, to dynamo action with a pronounced large-scale component. Cattaneo & Hughes (2006) took a somewhat different approach to the plane-layer convective dynamo, concentrating on more turbulent flows which, although rotationally influenced ($O(1)$ Rossby number) were not rotationally dominated; furthermore they considered horizontally extended domains, thereby allowing the convection to evolve without influence from sidewall boundaries. Their results differ considerably from those that emerge from considerations of 'well-ordered' convection. In particular they found that, despite the helical nature of the flow, two significant problems arose in determining α — even in the parameter regime where there is no self-sustaining dynamo action and hence $\mathcal{E}$ is due entirely to the imposed mean field. One problem is that α, even after spatial averaging over many convective cells, is a strongly fluctuating quantity in time (see Fig. 2.2); the second is that, even if one is bold enough to assign a mean value to α then this is extremely small — only $O(u/Rm)$ (where u is the rms velocity) rather than $O(u)$ as one might expect in a tur-

bulent process. The underlying problem — and one that seems inescapable
— is that at large *Rm* the fluctuations completely dominate the mean. This
result is potentially extremely bad news for mean field dynamo modelling,
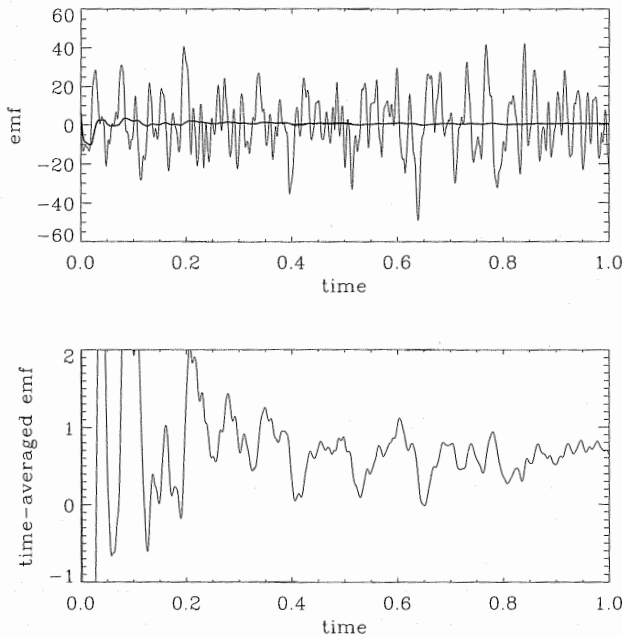and will be discussed further in Sec. 2.4.

Fig. 2.2   Time history of the spatially-averaged x-component of the emf for rotating
Boussinesq convection, with the rotation axis vertical and with an imposed magnetic
field of unit strength (measured in terms of the equivalent Alfvén velocity) in the x-
direction. Thus this component of the emf corresponds to α_{11}. The computational
domain has size $5 \times 5 \times 1$; the rms velocity $u \approx 50$ and $Rm \approx 250$. The spatial average
at any time involves many convective cells. Despite this, it can be seen that α is a wildly
fluctuating quantity in time. The thick line in the upper panel corresponds to the time
average up to time t of the signal. The same curve is plotted again in the lower panel
with a more appropriate vertical axis, showing a painfully slow convergence to a value
much smaller than that of u.

2.2.3   *Linear mean field dynamo models*

In the modelling of astrophysical magnetic fields it is natural to think of
the averaging procedure as an azimuthal average, and the mean fields that

result as therefore axisymmetric. In terms of cylindrical polar coordinates (s, ϕ, z) we may then write

$$\mathbf{U} = s\omega(s,z)\mathbf{e}_\phi + \mathbf{U}_p, \quad \mathbf{B} = B(s,z)\mathbf{e}_\phi + \mathbf{B}_p, \quad \boldsymbol{\mathcal{E}} = \mathcal{E}_\phi\mathbf{e}_\phi + \boldsymbol{\mathcal{E}}_p, \quad (2.22)$$

where the subscript p denotes the poloidal component. The quantities ω, $\mathbf{U}_p$ and $\mathbf{B}_p$ therefore represent the differential rotation, large-scale meridional flow and large-scale meridional field. For simplicity, but without losing the essence of mean field dynamo theory, we may consider the case of isotropic turbulence. The mean emf then takes the form

$$\boldsymbol{\mathcal{E}} = \alpha\mathbf{B} - \beta\nabla \times \mathbf{B}. \quad (2.23)$$

On writing $\mathbf{B}_p = \nabla \times A(s,z)\mathbf{e}_\phi$ (by axisymmetry), the mean induction equation (2.4) can be expressed as the two scalar equations:

$$\frac{\partial A}{\partial t} + s^{-1}\left(\mathbf{U}_p \cdot \nabla\right)(sA) = \alpha B + \eta_T\widetilde{\nabla}^2 A, \quad (2.24)$$

$$\frac{\partial B}{\partial t} + s\left(\mathbf{U}_p \cdot \nabla\right)\left(s^{-1}B\right) = s\left(\mathbf{B}_p \cdot \nabla\right)\omega + \left(\nabla \times \alpha\mathbf{B}_p\right)_\phi + \eta_T\widetilde{\nabla}^2 B, \quad (2.25)$$

where $\widetilde{\nabla}^2 = \nabla^2 - s^{-2}$, and where $\eta_T = \eta + \beta$ is the effective magnetic diffusivity (assumed constant in the above equations).

It can be seen from Eq. (2.24) that the α-effect can generate poloidal field from the toroidal component, and from Eq. (2.25) that it can also generate toroidal from poloidal field; the differential rotation ω can generate toroidal field by stretching out the poloidal field. A mean field dynamo must continually regenerate magnetic field through some combination of the α- and ω-effects; the relative magnitudes of these generation terms allows us to classify the nature of the dynamo action. If, on the right hand side of Eq. (2.25), the term involving α dominates that involving ω then we may neglect the $s\left(\mathbf{B}_p \cdot \nabla\right)\omega$ term and speak of an α^2-dynamo; if the ω term dominates we may neglect the $\left(\nabla \times \alpha\mathbf{B}_p\right)_\phi$ term and speak of an $\alpha\omega$-dynamo; if they are comparable then we speak of an $\alpha^2\omega$-dynamo. Note that the α-term in Eq. (2.24) is indispensable for mean field dynamo action. In its absence A simply decays; the removal of the source terms for B then leads to the eventual decay of the toroidal field also.

The key characteristics of these dynamos can be brought out in a Cartesian model, for which Eq. (2.24) and Eq. (2.25) simplify to

$$\frac{\partial A}{\partial t} + \left(\mathbf{U}_p \cdot \nabla\right)A = \alpha B + \eta_T\nabla^2 A, \quad (2.26)$$

$$\frac{\partial B}{\partial t} + \left(\mathbf{U}_p \cdot \nabla\right)B = \left(\mathbf{B}_p \cdot \nabla\right)U - \alpha\nabla^2 A + \eta_T\nabla^2 B, \quad (2.27)$$

where $\mathbf{U} = \mathbf{U}_p + U\mathbf{e}_y$, $\mathbf{B} = \mathbf{B}_p + B\mathbf{e}_y$. Under the assumption that $\mathbf{U}_p$, α and ∇U may be treated as uniform, we may seek local solutions of these equations, of the form

$$A = \tilde{A}\exp\left(pt + i\mathbf{k}\cdot\mathbf{x}\right), \qquad (2.28)$$

leading to the following dispersion relation governing the growth rate p:

$$\left(p + i(\mathbf{U}_p\cdot\mathbf{k}) + \eta k^2\right)^2 = \alpha^2 k^2 - i\alpha(\mathbf{k}\times\nabla U)_y \qquad (2.29)$$

(Parker 1955). Exponential growth of the field — a kinematic mean field dynamo — occurs if $\mathrm{Re}(p) > 0$.

For an α^2-dynamo with no meridional circulation ($\mathbf{U}_p = 0$), Eq. (2.29) simplifies to

$$p = \pm\alpha k + \eta k^2, \qquad (2.30)$$

showing that steady dynamo action ($\mathrm{Re}(p) > 0$, $\mathrm{Im}(p) = 0$) will occur provided $|\alpha| > \eta k$; i.e. growth is guaranteed for sufficiently small k.

For an $\alpha\omega$-dynamo with no meridional circulation, Eq. (2.29) reduces to

$$\left(p + \eta k^2\right)^2 = -i\alpha(\mathbf{k}\times\nabla U)_y = 2i\Gamma, \quad \text{say.} \qquad (2.31)$$

Dynamo action occurs if $\Gamma > \eta^2 k^4$; i.e. if the *dynamo number* $D = \Gamma/\eta^2 k^4$ is greater than unity. The growing solution takes the form of a travelling wave, propagating in the $+\mathbf{k}$ ($-\mathbf{k}$) direction if $\Gamma < 0$ ($\Gamma > 0$). (More generally it can be shown that for spatially varying underlying quantities, dynamo waves tend to propagate along lines of constant rotation (Yoshimura 1978).) If the equatorial migration of sunspots is viewed as the surface manifestation of an underlying, equatorially propagating, dynamo wave, then this suggests that in the northern (southern) hemisphere of the Sun the product $\alpha\partial\omega/\partial r$ should be negative (positive). From Eq. (2.29) it can be seen that, at least within this simple model, a meridional flow affects the frequency, but not the growth rate, of any dynamo, with the α^2-dynamo now also assuming an oscillatory character, and with the direction of propagation for the $\alpha\omega$-dynamo dependent (for $\Gamma > 0$) on the sign of $\Gamma^{1/2} - \mathbf{U}_p\cdot\mathbf{k}$.

Although all the fundamental ingredients of mean field dynamos are contained in the extremely simple plane layer model described above, clearly many important effects are omitted, even when attention is restricted to kinematic dynamos. Linear mean field dynamos in a sphere were considered by Steenbeck & Krause (1969a, b), Roberts & Stix (1972) and P.H. Roberts (1972). The evolution of dynamo waves in a spherical $\alpha\omega$-dynamo for one
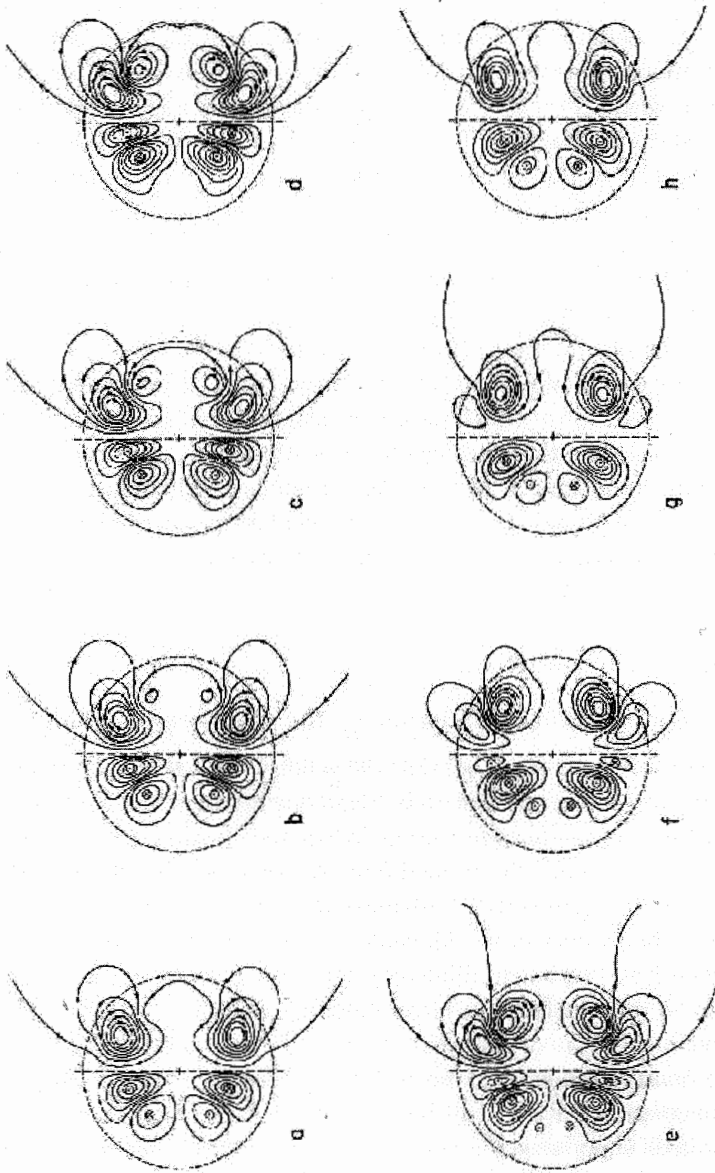
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Fig. 2.3   Evolution of the magnetic field through one cycle of a kinematic $\alpha\omega$-dynamo (from P.H. Roberts 1972). Meridian sections are shown, with poloidal field lines on the right and lines of constant toroidal field strength on the left. For this choice of parameters it can be seen that the pattern of the field progresses from the equator to the poles.

specific choice of α and ω is shown in Fig. 2.3. In certain astrophysical circumstances there are good reasons for believing that the α-effect and/or the ω effect are concentrated into regions that are thin in comparison with the body as a whole (for example, the solar tachocline, a thin region of radial velocity shear at the base of the convection zone, is widely conjectured to be the location of the ω-effect of the solar dynamo). Mathematically, such concentrated generation and shear can be modelled by δ-function representations of the α- and ω-effects. Away from the generation regions, A and B satisfy diffusion equations, with the generation terms brought in through the interface conditions; this approach has been investigated by Steenbeck & Krause (1966), Moffatt (1978) and Kleeorin & Ruzmaikin (1981). The role of a weakly modulated background state on the propagation of short wavelength dynamo modes was first considered by Kuzanyan & Sokoloff (1995), who showed that the eigenfunction of the magnetic field is localised away from the maximum of the dynamo number. The related problem of the influence of lateral boundaries (i.e. boundaries in the direction of propagation of the dynamo waves) has been considered in the linear regime by Worledge *et al.* (1997).

2.3   Incorporating Nonlinear Effects

2.3.1   *The nonlinear behaviour of the transport coefficients*

The discussion above indicates that the validity of mean field electrodynamics is unclear even in the kinematic regime, where the back-reaction of the Lorentz force on the plasma motions is ignored. In this section, we discuss the role of the nonlinearity in modifying the transport coefficients. We proceed on the (far-from-certain) assumption that mean field electrodynamics yields sensible results in the kinematic regime, with transport coefficients (the α-effect and turbulent diffusivity) of the right order of magnitude, and ask how the generation of a mean magnetic field affects the transport in the flow.

The question above may be restated as, 'What is the strength of the mean magnetic field that can be generated by the α-effect before the α-effect is itself suppressed via nonlinear effects in the momentum equation?' The momentum equation for an incompressible flow (with unit density) is given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla p + \mathbf{j} \times \mathbf{b} + \frac{1}{Re}\nabla^2 \mathbf{u} + \mathbf{F}, \qquad (2.32)$$

where $\mathbf{F}$ represents any external forces. The Lorentz force in a dynamo is

believed to be significant once the magnetic energy reaches the same order
of magnitude as the kinetic energy of the flow that is generating it. Within
the mean field framework this has traditionally been assumed to occur when
the magnetic energy of the mean field reaches equipartition with the kinetic
energy of the flow, i.e. when

$$\langle \rho u^2 \rangle \sim B_0^2/\mu_0. \tag{2.33}$$

Hence it is convenient to model the mean field transport coefficients as
having the dependence

$$\alpha = \frac{\alpha_0}{1 + B_0^2/\mathcal{B}^2}, \qquad \beta = \frac{\beta_0}{1 + B_0^2/\mathcal{B}^2}, \tag{2.34}$$

where α_0 and β_0 are the kinematic values of the transport coefficients and
$\mathcal{B}$ is the strength of the equipartition field. These, and other similar ex-
pressions, have been extensively used in models of astrophysical dynamos
(Jepps 1975; see the review by Ossendrijver 2003 for further references).
They are particularly appealing for a number of reasons. Clearly their
inclusion will lead to a model that is capable of generating a substantial
mean field — they are designed precisely to allow a mean field that has an
energy comparable with that of the turbulence. In addition, these formulae
are simple to implement. In the mean field ansatz *only* the mean field is
calculated, and so a nonlinearity that relies solely on the amplitude of the
mean field sits self-consistently within the framework of the theory.

There is, however, significant doubt over the validity of these expres-
sions at high Rm. Moreover, the uncertainties exist at such a fundamental
level that even the order of magnitude of the mean field generated in the
nonlinear regime is open to question. The issue of the level of saturation
of the mean magnetic field is contentious and subtle, and a comprehensive
discussion is well beyond the scope of this paper (see, for example, the re-
cent review by Diamond *et al.* 2005a). However, as the issue is of such
importance for astrophysical dynamo theory, a brief introduction will be
included here. Perhaps the clearest (and most convincing) argument for
the level of suppression of the transport coefficients is on physical grounds
(see, for example, Vainshtein &. Cattaneo 1992). It is generally accepted
(as argued above) that the kinematic phase in a mean field dynamo will
continue until the magnetic energy becomes comparable with that of the
turbulence. Moreover at high Rm, both theory and numerical experiments
indicate that the magnetic energy of the small-scale field will be much larger
than that of the mean field. Turbulence is known to amplify magnetic fields
locally (i.e. on the scale of the small-scale eddies with short turnover times)

so that the small-scale fields are more energetic than those on large scales — indeed most numerical experiments indicate that the ratio of the energy of the small-scale field to that of the large-scale field is given by

$$\langle b^2 \rangle \sim Rm^q B_0^2, \tag{2.35}$$

where $0 \leq q \leq 2$ is a flow- and geometry-dependent coefficient.

This, of course, poses a major problem for mean field theory. In particular, from this simple order of magnitude argument it appears as though the transport coefficients will be altered significantly once the strong *small-scale* field reaches equipartition with the kinetic energy of the turbulence, i.e. when

$$\langle \rho u^2 \rangle \sim Rm^q B_0^2/\mu_0. \tag{2.36}$$

(Though the dynamic effect of the small-scale field will depend also on its *filling factor* — i.e. on how much of the volume is occupied by strong small-scale fields.) If Eq. (2.36) rather than Eq. (2.33) is the correct estimate of the strength at which the large-scale field becomes dynamically significant, then the expressions in Eq. (2.34) should be replaced by

$$\alpha = \frac{\alpha_0}{1 + Rm^q B_0^2/\mathcal{B}^2}, \qquad \beta = \frac{\beta_0}{1 + Rm^q B_0^2/\mathcal{B}^2}. \tag{2.37}$$

This leads to what is sometimes referred to as 'catastrophic' quenching of the transport coefficients; i.e. a dramatic reduction in their efficiency even when the mean field is many orders of magnitude smaller than equipartition. The above argument relies on simple physical balances, but seems plausible nonetheless. However, calculation of the transport coefficients relies on correlations between the turbulence and the small-scale field, and these may have unexpected properties. There are currently two possible ways of discriminating between the expressions in Eq. (2.34) and Eq. (2.37), neither of which is entirely satisfactory. The first is to derive analytical expressions for α and β by applying conservation laws and closure approximations. The second is to compute straightforwardly the correlations between the small-scale fields at moderate Rm — and then to extrapolate the results to higher Rm.

Analytical progress may be made in certain circumstances by first recognising that the induction equation (and indeed Ohm's Law, which leads to its derivation) results in a number of relations which, in the absence of dissipation, reduce to conservation laws. These take the form (see Diamond

et al. 2005a; Brandenburg & Dobler 2001; Blackman & Field 2000)

$$\boldsymbol{\mathcal{E}} \cdot \mathbf{B}_0 = -\frac{1}{\sigma}\langle \mathbf{j} \cdot \mathbf{b}\rangle + \langle \mathbf{e} \cdot \mathbf{b}\rangle = \alpha B_0^2, \tag{2.38}$$

$$\frac{d\langle \mathbf{a} \cdot \mathbf{b}\rangle}{dt} = -2\langle \mathbf{e} \cdot \mathbf{b}\rangle + \langle \nabla \cdot (\mathbf{b}\phi)\rangle - \langle \nabla \times (\mathbf{a} \times \mathbf{e})\rangle, \tag{2.39}$$

$$\frac{dH_M}{dt} = -2\eta Q_H - F_H, \tag{2.40}$$

where ϕ is an arbitrary scalar (gauge) that is used to define the vector potential $\mathbf{A}$, H_M is the total magnetic helicity $\langle \mathbf{A} \cdot \mathbf{B}\rangle$, Q_H is the total current helicity $\langle \mathbf{J} \cdot \mathbf{B}\rangle$, and F_H is the surface-integrated magnetic helicity flux through the boundaries of the domain. The first of these equations follows simply from Ohm's Law, and relates α and the mean field to correlations between the small-scale magnetic fields, currents and emfs. The second is a conservation law for the averaged small-scale magnetic helicity — for statistically steady fields and suitable boundary conditions this may be simplified substantially to yield an equation for $\langle \mathbf{e} \cdot \mathbf{b}\rangle$ (see Diamond *et al.* 2005a). The final equation expresses conservation of magnetic helicity. In the absence of magnetic diffusion and for boundary conditions that do not allow magnetic helicity to leave the domain the total magnetic helicity is conserved. Hence, under these constraints, if the large-scale magnetic helicity is to grow it must be at the expense of the small-scale magnetic helicity. These relations all allow the achievement of a similar goal — they permit the expression of the average emf $\boldsymbol{\mathcal{E}}$, which relies on correlations between $\mathbf{u}$ and $\mathbf{b}$, in terms of correlations between $\mathbf{j}$ and $\mathbf{b}$. This substitution is not much more than a matter of taste, since *a priori* we know *none* of $\mathbf{u}$, $\mathbf{j}$ or $\mathbf{b}$ in the dynamic regime. However, the above expressions *do* become useful if they can be combined with a further expression relating α to $\langle \mathbf{b} \cdot \mathbf{j}\rangle$. As described below, this has been done, in an approximate sense, by Pouquet *et al.* (1976).

As stated above, in order to calculate α it is necessary to solve for the fluctuating magnetic field and velocity. The dynamics is all contained in the role of the Lorentz force in modifying the small-scale velocity, which at high Rm may be subtle and unexpected. A widely used, though often misunderstood, result utilises the eddy-damped quasi-normal Markovian (EDQNM) framework to approximate this back-reaction for turbulence possessing a short correlation time. This approach has been used extensively for purely hydrodynamic turbulence (see, for example, Frisch 1995; Davidson 2004), although the range of its applicability is still an open question. For magnetised flows, in which the turbulence at high Rm appears to deviate

significantly from being normally distributed, there is no convincing evidence that the theory is applicable in any regime. Nonetheless, in the limit of short correlation time, and assuming that the magnetic field has no effect on the correlation time of the turbulence, the EDQNM formalism yields a relationship between α and $\langle \mathbf{b} \cdot \mathbf{j} \rangle$, namely

$$\alpha = -\frac{\tau_c}{3} \langle \mathbf{u} \cdot \boldsymbol{\omega} - \mathbf{b} \cdot \mathbf{j} \rangle, \qquad (2.41)$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ (Pouquet *et al.* 1976). Note that in the kinematic limit this expression reduces to that given in Eq. (2.15), and that the effect of the Lorentz force is thus to add to the kinematic expression a magnetically-driven α-effect $\alpha_M = \tau_c/3 \langle \mathbf{b} \cdot \mathbf{j} \rangle$. There are two other ways to derive this approximate result; either by using closure mode such as the 'minimal τ approximation' (Kleeorin *et al.* 2002), or by linearising about a pre-existing turbulent MHD state with low Rm and short τ_c (Proctor 2003). (In the second approach, great care must be taken in the interpretation of the meaning of the variables $\mathbf{u}$, $\boldsymbol{\omega}$, $\mathbf{b}$ and $\mathbf{j}$, as noted by Proctor.) It is worth stressing again that this approach has circumvented the need to calculate correlations between $\mathbf{u}$ and $\mathbf{b}$ in favour of calculating those between $\mathbf{b}$ and $\mathbf{j}$ — clearly some dynamical relationship between $\mathbf{j}$ and $\mathbf{u}$ has been assumed (or approximated).

Ignoring for the minute any reservations about the validity of expression (2.41), it is now clear how to derive expressions for the dynamic effects of the magnetic field on the α-effect (and similarly for β). Combining one of the conservation laws with Eq. (2.41) yields either a dynamic result for the evolution of the magnetically-driven α-effect (α_M) (e.g. Blackman & Brandenburg 2002) or, under the assumption of stationarity, a simplified expression for α in the nonlinear regime — see, for example, the expressions in Gruzinov & Diamond (1994, 1995, 1996) and Kleeorin *et al.* (1995). In general these expressions support the notion that the α-effect *is* catastrophically quenched at high Rm (with the nature of the β-effect much less clear), though they indicate that the presence of a large-scale current ($\mathbf{J}$) or of boundary conditions that allow magnetic helicity to escape from the domain may lead to alleviation of the catastrophic quench.

The other approach to the question of the nonlinear suppression is to perform numerical simulations to calculate explicitly the emf as a function of Rm and B_0. This has the advantage that no approximations are necessary, but the disadvantage that results can only be obtained for fairly moderate values of Rm. Despite claims to the contrary in the literature, this is actually a reasonably severe restriction (as discussed below), and

considerable care should be taken in assessing the claimed values of Rm in different calculations. As an order of magnitude estimate, a calculation performed with 1000^3 spectral modes (or equivalently 1600^3 finite difference points), with a separation of scales of about a decade between domain size and the size of the most energetic turbulent eddy, is able accurately to reach magnetic Reynolds numbers of $O(10^3)$.

Two classes of numerical calculations have been performed in order to calculate the nonlinear dependence of the transport coefficients. The first employs a prescribed body force $\mathbf{F}(\mathbf{x}, t)$ in Eq. (2.32) in order to drive the flow $\mathbf{u}$. This can be carefully selected so as to drive a flow with particularly advantageous properties for large-scale dynamo action (in a similar manner to the way kinematic flows are selected to be maximally helical so that they yield strong α-effects). The forcing is typically chosen in order to drive a flow with large pointwise and net helicity and also strong time-dependence, so that particle trajectories within the flow are chaotic. Separation of scales can be achieved in two ways. One is to force the flow on a scale much smaller than that of the computational domain; e.g. in a spectral calculation applying the forcing at a wavenumber $k_f = 8$, which determines the small-scale, and ascribing flows and fields with $k = 1$ to be large-scale (e.g. Brandenburg 2001). Here, however, there is no natural separation of scales and a continuous spectrum of scales emerges. The other (Cattaneo & Hughes 1996) is to drive the small-scale flow on the scale of the computational domain ($k_f = 1$) and to achieve the separation of scales by imposing an infinite-scale ($k = 0$) mean magnetic field ($\mathbf{B}_0$), as in the kinematic case described above.

The second approach to calculating the α-effect numerically in the nonlinear regime is to drive the flow via the action of buoyancy in a rotating convective flow (either compressible or incompressible), as discussed for the kinematic case in Sec. 2.2.2. A mean magnetic field is inserted and the emf $\boldsymbol{\mathcal{E}}$ (and hence α) calculated self-consistently, taking account of the Lorentz force.

There are now many examples of calculations employing both these approaches (e.g. Cattaneo & Hughes 1996; Cattaneo *et al.* 2002). Here we proceed by giving details of one such calculation (Cattaneo & Hughes 1996) and then discuss its relation to other such calculations later in the section. Here, the forcing $\mathbf{F}(\mathbf{x}, t)$ is chosen so that in the absence of a magnetic field the Galloway-Proctor flow Eq. (2.21) is driven. A mean magnetic field is then imposed in the z-direction and the full magnetohydrodynamic system (Eq. (2.2) and Eq. (2.32)) evolved to a statistically steady state. The components of the mean emf $\boldsymbol{\mathcal{E}} = (\mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z)$ are then calculated as a
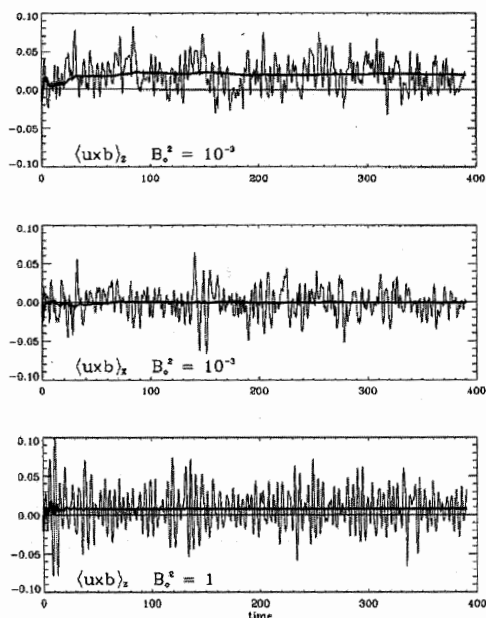
Fig. 2.4   Time histories of components of the emf for the 3d turbulent flow driven by the Galloway-Proctor forcing with a magnetic field imposed in the z-direction (from Cattaneo & Hughes 1996). The emf has strong temporal fluctuations, but a meaningful time average (thick line) can clearly be defined. The top two panels show $\mathcal{E}_z$ and $\mathcal{E}_x$ for a weak magnetic field; the time average of $\mathcal{E}_x$ converges to zero, as expected from symmetry considerations. The lowest panel shows $\mathcal{E}_z$ for a field of almost equipartition strength.

function of Rm and $\mathbf{B}_0$, as shown in Fig. 2.4. These time series show clearly that the (space-averaged) emf is a wildly fluctuating quantity and that, in order to achieve average values for α that have converged to a meaningful result, temporal averages over significant time periods (very many turnover times of the flow) are necessary. However it *is* possible to integrate the equations for long enough to obtain a meaningful average value for α. This is plotted as a function of B_0 (at fixed Rm) and Rm (at three representative values of B_0) in figures 2.5a and 2.5b respectively. Taken together, these results strongly suggest that the α-effect is suppressed with a very strong dependence on field strength at high Rm. In particular, it appears possible to discriminate between the formulae in Eq. (2.34) and Eq. (2.37), with the numerical results indicating that Eq. (2.37) is a considerably more accurate description of the behaviour of the α-effect in the nonlinear regime.
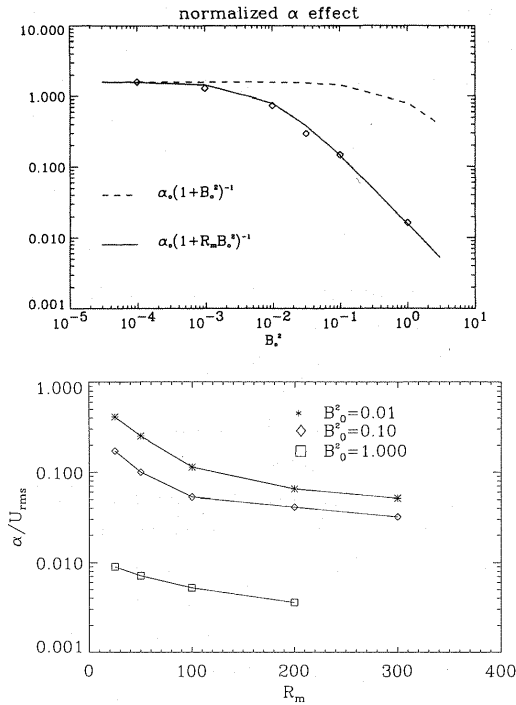
Fig. 2.5 α versus B_0^2 (upper panel, from Cattaneo & Hughes 1996) and versus Rm (lower panel, from Cattaneo *et al.* 2002), for the same flow as in Fig. 2.4. The upper panel shows the predictions of the two quenching formulae Eq. (2.34) and Eq. (2.37) (with $q = 1$), and shows that the numerical simulations strongly support the latter.

Many other numerical calculations with various assumptions and interpretations (e.g. Brandenburg 2001; Ossendrijver *et al.* 2003) also indicate that the transport coefficients may be catastrophically quenched in the nonlinear regime. These results have, however, been criticised on the grounds that they are all undertaken in closed domains with boundary conditions that do not allow magnetic helicity to leave the computational domain (Blackman & Field 2000). The argument put forward is that large-scale helicity can only grow at the expense of small-scale helicity, as discussed above, and that the system is thus overly constrained. This is an interesting point and one that merits further investigation. It is difficult however to envisage a situation where the flux of magnetic helicity out of the domain can alleviate the constraint on the transport coefficients without removing flux from the domain and limiting the efficiency of the dynamo. Moreover,

the physical argument for strong suppression outlined above relies solely on the dynamics of the small scales and it is not clear how the modification of boundary conditions (possibly far away from the turbulent small-scale eddies) can lead to a significant enhancement of the local transport properties of the turbulence.

2.3.2   *Other nonlinear effects*

The suppression of the transport coefficients in the nonlinear regime is contentious and is likely to remain so for the foreseeable future. This issue lies at the very heart of mean field electrodynamics as it determines the level of saturation of nonlinear mean field models. However, there are other nonlinear processes that *may* be of importance for saturating the exponential growth of field in a kinematic mean field model. In Sec. 2.2.3 we described how differential rotation is a key feature of the dynamo process, stretching out a poloidal field and converting it into toroidal field (the ω-effect). Therefore a possible saturation mechanism involves the modification of the differential rotation by the Lorentz force. In order to model this, a detailed understanding of the mechanism for the generation and maintenance of differential rotation by rotating turbulent motions is required. In general, mean flows are set up in turbulent rotating systems via small-scale correlations, leading to Reynolds stresses that naturally transport angular momentum, allowing the formation of a pattern of differential rotation. As in mean field electrodynamics, the role of correlations in this picture is crucial — and poorly understood (see Diamond *et al.* 2005b for a comprehensive review of the generation of zonal flows in plasmas, and Brummell *et al.* 1998 or Brun *et al.* 2004 for a discussion of the generation of mean flows and differential rotation by convection). In the mean field formalism, the generation of mean flows by small-scale correlations is often parameterised using the Λ-effect, a closure scheme that relates the average Reynolds stresses to the local angular velocity and its derivatives (see Rüdiger 1989 for a complete discussion). It is sufficient here to state that the theory underpinning the Λ-effect is a two-scale theory in a similar vein to mean field electrodynamics — and so has the same strengths and weaknesses.

Just as for the transport coefficients α and β, the role of magnetic fields in modifying angular momentum transport, and hence differential rotation, is also poorly understood. However what *is* known is that the magnetic field can modify the differential rotation in a straightforward manner, with

a large-scale magnetic field $\mathbf{B}_0$ driving a large-scale flow via the action of the large-scale Lorentz force $\mathbf{J}_0 \times \mathbf{B}_0$ in the momentum equation. This mechanism is often termed the 'Malkus-Proctor effect' (Malkus & Proctor 1975) in astrophysical dynamo theory (especially in solar dynamo theory, where this saturation mechanism has been extensively invoked). In addition, in a high Rm environment the presence of a large-scale field and a small-scale flow inevitably implies the presence of a strong small-scale field, as argued above. This small-scale field can itself influence the angular momentum transport in two ways; either by modifying the dynamics of the small-scale velocity (in particular the correlations that lead to the driving of mean flows in the first place) via the small-scale Lorentz force, or by itself driving mean flows via average Maxwell stresses (which arise through small-scale/small-scale interactions of magnetic fields). This process in mean field models has usually been either crudely parameterised in the form of ω-quenching (e.g. Roald & Thomas 1997), or Λ-quenching (e.g. Kitchatinov *et al.* 1994) in which the back-reaction of the magnetic field on the angular momentum transport is calculated within the same framework as the mean field equations. It is though important to note that all the difficulties and uncertainties surrounding the transport coefficients in mean field electrodynamics arise *a fortiori* in any theory of the modification of angular momentum transport.

2.3.3 *Nonlinear mean Field Models*

We discussed briefly in Sec. 2.2.3 how the framework of mean field electrodynamics could be used to construct kinematic dynamo models. As for the kinematic theory, most of the computational effort in the nonlinear regime has been 'object specific', with computations being undertaken with the specific aim of explaining the generation of magnetic fields in either planets, stars or galaxies. In these cases, plausible profiles for the differential rotation, α-effect and turbulent diffusivity are selected for the object to be modelled and, likewise, a nonlinear saturation mechanism is selected that may be of importance for the dynamo operating in that object. In this paper we do not attempt to discuss these astrophysically-motivated mean field models, as it is extremely difficult to draw robust conclusions from studies that aim specifically to model a particular astrophysical object in detail. Instead we focus on aspects of nonlinear behaviour that are inherent to all nonlinear mean field models, and refer the interested reader to reviews by Ossendrijver (2003) and Tobias & Weiss (2007), who discuss solar and

stellar models, by Jones (2003), who discusses planetary dynamos, and by Beck *et al.* (1996), who discuss the galactic dynamo.

2.3.3.1   *Nonlinear Travelling Waves*

Here we give a brief discussion of the role of nonlinearities in modifying the behaviour of the kinematic mean field dynamos discussed in Sec. 2.2.3 — i.e. plane-wave dynamos, bounded dynamos and spherical dynamos. The simplest example of a mean field dynamo (as discussed earlier) is the local travelling-wave solution of Parker (1955). There the solution is assumed to be independent of the z-coordinate and wave-like solutions depending on the x-coordinate are constructed. In order to extend this model to the nonlinear regime, where computational techniques are required, a finite domain must be considered. In two dimensions, boundary conditions must be imposed in both directions. Here we choose appropriate matching conditions in the z-direction and periodic boundary conditions in the x-direction (see Tobias 1997 for a detailed discussion). Within this framework it is now permissible to introduce the possibility that the transport coefficients (α and β) and the shear have an underlying dependence on the z-coordinate (though they remain independent of x). Such a model is local in the x-direction, but global in the z-direction. The nonlinear dynamics of such a model was investigated in detail in Tobias (1997) and the results summarised here.

The linear theory (as for the completely localised Parker model described earlier) indicates that as the dynamo number is increased past a threshold value, instability sets in to travelling wave solutions, with a well-defined preferred wavelength and frequency (which depend on the critical dynamo number). The waves now possess non-trivial spatial structure in the z-direction, determined by the underlying profiles of the transport coefficients. What happens to the nonlinear solutions as the dynamo number D is increased further depends on the form of the nonlinearity adopted.

The first class of nonlinearities considered are 'static nonlinearities' (such as α, β or ω-quenching), in which the mean magnetic field acts back instantaneously on the transport coefficient or differential rotation through a parameterised formula such as Eq. (2.34). For these instantaneous nonlinear effects the travelling wave solutions that are destabilised in the initial bifurcation remain stable as the dynamo number is increased and no further temporal structure is added. It is also found that α- and ω-quenching have similar properties, acting purely as equilibration mechanisms that have

little effect on the spatial dependence of the waves. Furthermore, for these cases, the wave-speed remains largely independent of the amplitude of the magnetic field as the dynamo number is increased. For β-quenching both the spatial structure and the wave speed of the travelling waves are modified in the nonlinear regime. Waves tend to travel more slowly as the magnetic diffusivity is quenched, and the dynamo becomes more inefficient. This change in the frequency of the waves due to diffusivity quenching may provide a constraint for oscillatory dynamo models.

In the second class of nonlinearities the magnetic field acts back via a separate dynamical equation. This could be due to the mean field driving large-scale flows directly (the Malkus-Proctor effect), or to the mean magnetic field suppressing the small-scale turbulence dynamically (which may involve the addition of a dynamic equation for the evolution of the transport coefficients, α, β or Λ). If the dynamic Malkus-Proctor effect is included as a nonlinearity, then the travelling wave solutions created in the initial bifurcation rapidly lose stability to states with more spatio-temporal variability. In particular, a complicated series of bifurcations, involving a secondary and tertiary Hopf bifurcation to modulated travelling waves and quasi-periodic waves, leads to a transition to chaotic oscillations. The basic magnetic cycle becomes modulated on a time-scale that is determined by the ratio of the turbulent diffusivities for magnetic field (β) and angular velocity (ν_T). This natural modulation of the basic magnetic cycle is often used as an explanation of intermittent behaviour in nonlinear dynamos — for example, the solar cycle is modulated and undergoes recurrent periods of reduced activity known as Grand Minima.

2.3.3.2 *Nonlinear Dynamos in Finite Cartesian and Spherical Domains*

The behaviour of nonlinear travelling waves described above is instructive, and yields some important results for the dependence of the amplitude and period of the mean field as a function of dynamo number. It also gives an indication of the role of dynamical nonlinearities in producing complicated spatio-temporal behaviour. However, we noted at the end of Sec. 2.2.3 that, even in the kinematic regime, dynamo solutions may have very different properties in finite domains to those found for periodic boundary conditions. Of particular importance is the interaction of the dynamo solutions with inhomogeneities in the underlying profiles of either the transport coefficients or the differential rotation, or the interaction with

boundaries. In the linear regime, both the frequency and the spatial form of the dynamo solutions were found to be sensitive to the imposition of boundary conditions and the inhomogeneities. It is therefore of no surprise that the behaviour of nonlinear mean field dynamo models in finite domains such as spheres, spherical shells, discs and Cartesian slabs may be very different to that of nonlinear travelling wave solutions.

As for the kinematic case, most of the calculations of nonlinear mean field models in finite domains are targeted at modelling dynamos in specific astrophysical objects, with the structure of the transport coefficients and rotation profiles usually chosen using some plausible physical assumptions for the object to be modelled. In a similar manner the nonlinearity is chosen in a plausible, but *ad hoc*, manner. There are therefore a vast number of nonlinear mean field dynamo calculations, whose conclusions differ owing to the different assumptions behind the model in question (see, for example, the reviews cited above by Beck *et al.* 1996; Jones 2003; Ossendrijver 2003; Tobias & Weiss 2007). Discriminating between such models is extremely difficult and largely subjective. We prefer here to focus on generic properties of nonlinear models in finite domains — i.e. properties that appear robust whichever assumptions are included in the model.

For nonlinear dynamo models in finite domains, the following properties tend to be observed. In general, the amplitude of the energy in the mean magnetic field is an increasing function of dynamo number. More specifically, the magnetic energy is often observed initially to increase linearly with supercriticality (i.e. $\langle \mathbf{B} \rangle^2 \propto (D - D_c)$) for D sufficiently close to its critical value D_c. As $D - D_c$ is increased further there is a saturation of the amplitude of the mean field as a function of dynamo number. More-over, as $D - D_c$ increases, the dynamo solution becomes more irregular for all types of nonlinearities used. A sequence of bifurcations leads to the spatio-temporal fragmentation of solutions and eventually to chaotic time dependence of the dynamo solutions. This pattern of spatio-temporal chaos may in general be the result of either nonlinear interactions between the magnetic field and the mean flow (as discussed above for nonlinear dynamo waves), or the interaction between mean magnetic modes with different symmetry properties. These interactions can be understood with reference to the theory of nonlinear dynamical systems (see, for example, Knobloch *et al.* 1998, Tobias 2002 for an in-depth discussion of the mechanisms that can lead to the formation of chaotic solutions). For all nonlinearities, the primary frequency of oscillation of the dynamo appears to be an increasing function of dynamo number. This is due to the interaction of the nonlinear

wavetrain with the boundaries of the domain, and the subsequent change in lengthscale for the dynamo wavetrain (see, for example, Tobias 1998).

Despite some common themes running through the nonlinear development of mean field dynamo models, there is still great uncertainty as to the range of applicability of such models. We stress again here that these models are based upon a theory for which even the kinematic behaviour is uncertain. It is clear that one must proceed with great care in evaluating the results of such nonlinear mean field models, though they may yet provide some insight into the mechanisms for the generation of magnetic field in astrophysical bodies.

2.4 Discussion

Since the formulation of mean field electrodynamics by Steenbeck, Krause and Rädler in the 1960's — foreshadowed by the work of Parker (1955) — it has been used extensively in the modelling of the generation and maintenance of magnetic fields in planets, stars, galaxies and accretion discs. It is, in some sense, a remarkably successful theory, being able to reproduce temporal and spatial features of a tremendous range of observed astrophysical magnetic fields. This suggests strongly therefore that the 'true' equation describing the evolution of the mean field does indeed take the form of Eq. (2.9) — although this is possibly not all that surprising, since the α-effect closes the dynamo loop and β is (in its simplest form) a turbulent diffusivity. However, that is not to say that the all is well with mean field electrodynamics. Even at the level of reproducing observed cosmical fields, the freedom in the choice of parameters means that models inspired by different physical assumptions can lead to the same global features. Thus it is not possible to assert that specific magnetic behaviour (such as the cyclic evolution of a star's magnetic field, for example) results from specific forms of α_{ij} and β_{ijk}, and hence from specific physical considerations. Conversely it is possible to have two rather similar mean field models that produce very different behaviour. Thus great care needs to be taken in the description — and, *a fortiori*, in the prediction — of astrophysical magnetic fields via parameterised mean field models.

Detailed investigation of the micro-physics of the mean field transport coefficients leads to further difficulties. Although for turbulent flows with low magnetic Reynolds numbers or very short correlation times the mean field description seems to hold, for the astrophysically relevant case of

turbulence with $Rm \gg 1$ and $O(1)$ correlation times the very foundations of mean field electrodynamics seem unsure. Here it seems unavoidable that the fluctuations in the magnetic field overwhelm the mean. This leads, as discussed earlier, to problems both in the kinematic regime, in which it can be hard even to determine α, and in the dynamic regime, where any α-effect is 'catastrophically' quenched by an extremely weak mean magnetic field. It appears that maybe what is required are less turbulent, more ordered flows. In the Sun these may arise from the presence of the tachocline, a thin shear region at the base of the convection zone. However, what happens in fully convective stars, for example, or with the evolution of galactic magnetic fields, remains very much an open question.

References

[1] Backus G E 1958 *Ann. Phys.* **4** 372–447.
[2] Backus G, Parker R and Constable C 1996 *Foundations of Geomagnetism* (Cambridge University Press, Cambridge).
[3] Balbus S A and Hawley J F 1998 *Rev. Mod. Phys.* **70** 1–53.
[4] Beck R, Brandenburg A, Moss D, Shukurov A and Sokoloff D 1996 *Ann. Rev. Astron. Astrophys.* **34** 155–6.
[5] Blackman E G and Brandenburg A 2002 *Astrophys. J.* **579** 359–73.
[6] Blackman E G and Field G B 2000 *Astrophys. J.* **534** 984–8.
[7] Brandenburg A 2001 *Astrophys. J.* **550** 824–40.
[8] Brandenburg A and Dobler W 2001 *Astrophys. J.* **369** 329–38.
[9] Brandenburg A and Subramanian K 2005 *Phys. Rep.* **417**, 1–209.
[10] Brummell N H, Hurlburt N E and Toomre J 1998 *Astrophys. J* **493** 955–69.
[11] Brun A S, Miesch M S and Toomre J 2004 *Astrophys. J* **614** 1073–1098.
[12] Cattaneo F and Hughes D W 1996 *Phys. Rev. E* **54** R4532–5.
[13] — 2006 *J. Fluid Mech.* **553**, 401–18.
[14] Cattaneo F, Hughes D W and Proctor 1988 *Geophys. Astrophys. Fluid Dyn.* **41** 335–42.
[15] Cattaneo F, Hughes D W and Thelen J-C 2002 *J. Fluid Mech.* **456** 219–37.
[16] Childress S 1979 *Phys. Earth Planet. Inter.* **20** 172–80.
[17] Childress S and Gilbert A D 1995 *Stretch, Twist, Fold: The Fast Dynamo* (Springer-Verlag, Berlin).
[18] Childress S and Soward A M 1972 *Phys. Rev. Lett.* **29** 837–9.
[19] Childress S and Soward A M 1989 *J. Fluid Mech.* **205** 99–133.
[20] Courvoisier A, Hughes D W and Tobias S M 2006 *Phys. Rev. Lett.* 034503.
[21] Cowling T G 1934 *Mon. Not. R. Astr. Soc.* **94** 39–48.
[22] Davidson P A 2004 *Turbulence: an Introduction for Scientists and Engineers* Oxford University Press: Oxford.
[23] Diamond P H, Hughes D W and Kim E 2005a In *Fluid Dynamics and Dynamos in Astrophysics and Geophysics* (eds A M Soward, C A Jones,

D W Hughes and N O Weiss) (CRC press, Boca Raton) p 145–92.

[24] Diamond P H, Itoh S-I, Itoh K and Hahm T S 2005b *Plasma Phys. Control. Fusion* **47** R35–R161.

[25] Frisch U 1995 *Turbulence: the Legacy of A N Kolmogorov* (Cambridge University Press, Cambridge).

[26] Galloway D J and Proctor M R E 1992 *Nature* **356** 691–3.

[27] Gruzinov A V and Diamond P H 1994 *Phys. Rev. Lett.* **72** 1651–4.

[28] Gruzinov A V and Diamond P H 1995 *Phys. Plasmas* **2** 1941–6.

[29] Gruzinov A V and Diamond P H 1996 *Phys. Plasmas* **3** 1853–7.

[30] Herzenberg A 1958 *Phil. Trans. R. Soc. Lond.* A**250** 543–83.

[31] Hide R and Palmer T N 1982 *Geophys. Astrophys. Fluid Dyn.* **19** 301–9.

[32] Ivers D J and James R W 1984 *Phil. Trans. R. Soc. Lond.* A**312** 179–218.

[33] Jepps S A 1975 *J. Fluid Mech.* **67** 625–45.

[34] Jones C A 2003 In *Stellar Astrophysical Fluid Dynamics* (eds. M J Thompson and J Christensen-Dalsgaard) (Cambridge University Press, Cambridge) p 159–176.

[35] Jones C A and Roberts P H 2000 *J. Fluid Mech.* **404** 311–43.

[36] Kitchatinov L L, Rüdiger G and Küker M 1994 *Astron. Astrophys.* **292** 125–32.

[37] Klapper I and Young L S 1995 *Commun. Math. Phys.* **173** 623–46.

[38] Kleeorin N and Ruzmaikin A A 1981 *Geophys. Astrophys. Fluid Dyn.* **17** 281–96.

[39] Kleeorin N, Rogachevskii I and Ruzmaikin A 1995 *Astron. Astrophys.* **297** 159–167.

[40] Kleeorin N, Rogachevskii I and Sokoloff D 2002 *Phys. Rev. E* **65** 36303-7.

[41] Knobloch E, Tobias S M and Weiss N O 1998 *Mon. Not. R. Astr. Soc.* **297** 1123–38.

[42] Krause F and Rädler K-H 1980 *Mean Field Electrodynamics and Dynamo Theory* (Pergamon Press, Oxford).

[43] Kuzanyan K M and Sokoloff D D 1995 *Geophys. Astrophys. Fluid Dyn.* **81** 113–29.

[44] Larmor J 1919 *Rep. Brit. Assoc. Adv. Sci. 1919* 159–60.

[45] Malkus W V R and Proctor M R E 1975 *J. Fluid Mech.* **67** 417–44.

[46] Moffatt H K 1974 *J. Fluid Mech* **65** 1–10.

[47] Moffatt H K 1978 *Magnetic field generation in electrically conducting fluids* (Cambridge University Press, Cambridge).

[48] Moffatt H K 1983 *Rep. Prog. Phys.* **46** 621–64.

[49] Ossendrijver M 2003 *Astron. Astrophys. Rev.* **11** 287–367.

[50] Parker E N 1955 *Astrophys. J.* **122** 293–314.

[51] Parker E N 1979 *Cosmical Magnetic Fields: Their Origin and Their Activity* (Clarendon Press, Oxford).

[52] Plunian F and Rädler K-H 2002 *Geophys. Astrophys. Fluid Dyn.* **96** 115–33.

[53] Pouquet A, Frisch U and Léorat J 1976 *J. Fluid Mech.* **77** 321–54.

[54] Proctor M R E 2003 In *Stellar Astrophysical Fluid Dynamics* (eds. M J Thompson and J Christensen-Dalsgaard) (Cambridge University Press, Cambridge) p 143–158.

[55] Roald C B and Thomas J H 1997 *Mon. Not. R. Astr. Soc.* **288** 551–64.

[56] Roberts G O 1972 *Phil. Trans. R. Soc. Lond.* A**271** 411–54.

[57] Roberts P H 1972 *Phil. Trans. R. Soc. Lond.* A**272** 663–98.

[58] Roberts P H and Soward A M 1992 *Ann. Rev. Fluid Mech.* **24** 459–512.

[59] Roberts P H and Stix M 1972 *Astron. Astrophys.* **18** 453–66.

[60] Rosner R 2000 *Phil. Trans. R. Soc. Lond.* A**358** 689–709.

[61] Rotvig J and Jones C A 2002 *Phys. Rev. E* **66** 056308-1–15.

[62] Rüdiger G 1989 *Differential Rotation and Stellar Convection: Sun and Solar-Type Stars* (Gordon and Breach, New York).

[63] Ruzmaikin A A, Shukurov A M and Sokoloff D D 1988 *Magnetic Fields of Galaxies* (Kluwer, Dordrecht).

[64] Soward A M 1974 *Phil. Trans. R. Soc. Lond.* A**275** 611–46.

[65] Soward A M 1987 *J. Fluid Mech.* **180** 267–95.

[66] Steenbeck M and Krause F 1966 *Z. Naturforsch.* **21**a 1285–96.

[67] Steenbeck M and Krause F 1969a *Astron. Nachr.* **291** 49–84.

[68] Steenbeck M and Krause F 1969b *Astron. Nachr.* **291** 271–86.

[69] Steenbeck M, Krause F and Rädler K-H 1966 *Z. Naturforsch.* **21**a 369–76.

[70] Stellmach S and Hansen U 2004 *Phys. Rev. E* **70** 056312-1–16.

[71] Tobias S M 1997 *Geophys. Astrophys. Fluid Dyn.* **86** 287–343.

[72] Tobias S M 1998 *Mon. Not. R. Astr. Soc.* **296** 653–61.

[73] Tobias S M 2002 *Astron. Nachr.* **323** 417–423.

[74] Tobias S M & Weiss N O 2007 In *The Solar Tachocline* (eds D W Hughes, R Rosner and N O Weiss) (Cambridge University Press, Cambridge) p319–50.

[75] Vainshtein S I and Cattaneo F 1992 *Astrophys. J.* **393** 165–71.

[76] Vishik M M 1989 *Geophys. Astrophys. Fluid Dyn.* **48** 151–67.

[77] Worledge D, Knobloch E, Tobias S and Proctor M 1997 *Proc. R. Soc. Lond.* A**453** 119–43.

[78] Yoshimura H 1978 *Astrophys. J.* **226** 706–19.

[79] Zeldovich Ya B 1957 *Sov. Phys. JETP* **4** 460–2.

[80] Zweibel E G 2005 In *Fluid Dynamics and Dynamos in Astrophysics and Geophysics* (eds A M Soward, C A Jones, D W Hughes and N O Weiss) (CRC press, Boca Raton) p115-43.