Solutions of nonlinear stochastic differential equations with long-range power-law distributions

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$1/f$ noise

Our research is related to the 1/f noise problem and long-range processes.

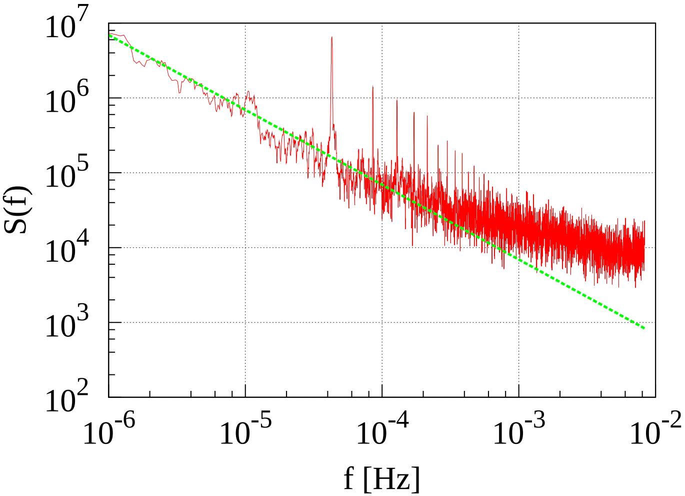
$1/f$ noise

a type of noise whose power spectral density $S(f)$ behaves like

$$S(f) \sim 1/f^{\beta}\,, \qquad \beta \text{ is close to } 1$$

Fluctuations of signals exhibiting $1/f$ behavior of the power spectral density at low frequencies have been observed in a wide variety of physical, geophysical, biological, financial, traffic, Internet, astrophysical and other systems.

Example of $1/f$ noise



Power spectral density of trading activity (number of trades per 1 min). for ABT stock on NYSE

Notes about $1/f$ noise

- A pure $1/f$ power spectrum is physically impossible because the total power would be infinity.
- We search for a model where the spectrum of signal has $1/f^{\beta}$ behavior only in some intermediate region of frequencies, $f_{\mathrm{min}} \ll f \ll f_{\mathrm{max}}$, whereas for small frequencies $f \ll f_{\mathrm{min}}$ the spectrum is bounded.
- The behavior of spectrum at frequencies $f_{\mathrm{min}} \ll f \ll f_{\mathrm{max}}$ is connected with the behavior of the autocorrelation function at times $1/f_{\mathrm{max}} \ll t \ll 1/f_{\mathrm{min}}$.

Notes about $1/f$ noise

- Often $1/f$ noise is defined by a long-memory process, characterized by $S(f) \sim 1/f^{\beta}$ as $f \to 0$.
- This long-range dependence property is equivalent to similar behavior of autocorrelation function $C(t)$ as $t \to \infty$
- This behavior of the autocorrelation function is not necessary for obtaining required form of the power spectrum in a finite interval of the frequencies which does not include zero

$1/f$ noise

- $1/f$ noise is intermediate between white noise, $S(f) \sim 1/f^0$ and Brownian motion $S(f) \sim 1/f^2$
- In contrast to the Brownian motion generated by the linear stochastic equations, the signals and processes with $1/f$ spectrum cannot be understood and modeled in such a way.

Goal

to find a simple nonlinear stochastic differential equation (SDE) generating signals exibiting $1/f$ noise

Application to the description of trading

- Time series of financial data exhibit highly nontrivial statistical properties. Many of these properties appear to be universal.
- Trading activity, trading volume, and volatility are stochastic variables with the long-range correlation. The autocorrelation of the volatility decays only slowly as a power law.
- Probability distribution functions (PDFs) of return and trading activity have fat tails exhibiting power-law decay.
- Proposed equations can exhibit both power-law PDF and power-law spectrum.

Justification of SDE

- If $S(f) \sim f^{-\beta}$ then power spectral density has a scaling property

$$S(af) = a^{-\beta} S(f)$$

- Wiener-Khintchine theorem

$$C(t) = \int_{-\infty}^{+\infty} S(f) \cos(2\pi f t)\, \mathrm{d}f$$

- Autocorrelation function $C(t)$ has scaling property

$$C(at) \sim a^{\beta-1} C(t)$$

Justification of SDE

- Autocorrelation function can be written as

$$C(t) = \int \mathrm{d}x \int \mathrm{d}x' \, xx' P_0(x) P_x(x', t|x, 0)$$

- $P_0(x)$ is the steady state PDF
- $P_x(x', t|x, 0)$ is the transition probability
- The transition probability can be obtained from the solution of the Fokker-Planck equation with the initial condition $P_x(x', 0|x, 0) = \delta(x' - x)$.

Justification of SDE

- Let us assume that
  - Steady state PDF has power-law form

$$P_0(x) \sim x^{-\nu}$$

  - Trasnsition probability has a scaling property

$$P(ax', t|ax, 0) = a^{-1} P(x', a^{2(\eta-1)} t|x, 0)$$

- Then the autocorrelation function will have the required scaling with

$$\beta = 1 + \frac{\nu - 3}{2(\eta - 1)}$$

Justification of SDE

To get the required scaling of transition probability:

- SDE will contain only powers of x
- The diffusion coefficient will be of the form $x^{2\eta}$
- The drift term is fixed by the requirement that the steady-state PDF should be $x^{-\nu}$

Proposed SDE

$$\mathrm{d}x = \sigma^2(\eta - \nu/2)x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W_t$$

B. Kaulakys and J. Ruseckas, Phys. Rev. E **70**, 020101(R) (2004).

B. Kaulakys and J. Ruseckas, V. Gontis, and M. Alaburda, Physica A **365**, 217 (2006).

Restriction of diffusion

- Because of the divergence of the power-law distribution and the requirement of the stationarity of the process, the SDE should be analyzed together with the appropriate restrictions of the diffusion in some finite interval.
- When diffusion is restricted, scaling properties are only approximate, but $1/f$ spectrum remains in a wide interval of frequencies.

Restriction of diffusion

Possible forms of restriction:

- Reflective boundary conditions at $x = x_{\min}$ and $x = x_{\max}$
- Exponential restriction of the diffusion

$$\mathrm{d}x = \sigma^2 \left(\eta - \frac{\nu}{2} + \frac{m}{2} \left(\frac{x_{\min}}{x} \right)^m - \frac{m}{2} \left(\frac{x}{x_{\max}} \right)^m \right) x^{2\eta - 1} \mathrm{d}t + \sigma x^\eta \mathrm{d}W_t$$

Steady state PDF:

$$P_0(x) \sim x^{-\nu} \exp \left(- \left(\frac{x_{\min}}{x} \right)^m - \left(\frac{x}{x_{\max}} \right)^m \right)$$

Restriction of diffusion

- q-exponential steady-state PDF

$$\mathrm{d}x = \sigma^2(\eta - \nu/2)(x + x_0)^{2\eta-1}\mathrm{d}t + \sigma(x + x_0)^{\eta}\mathrm{d}W_t$$

$$P_0(x) \sim \exp_{1+1/\nu}(-\nu x/x_0)$$

Reflective boundary condition at $x = 0$

B. Kaulakys and M. Alaburda, J. Stat. Mech. **2009**, P02051 (2009).

- q-Gaussian steady-state PDF

$$\mathrm{d}x = \sigma^2(\eta - \nu/2)(x^2 + x_0^2)^{\eta-1}x\mathrm{d}t + \sigma(x^2 + x_0^2)^{\eta/2}\mathrm{d}W_t$$

$$P_0(x) \sim \exp_{1+2/\nu}(-\nu x^2/2x_0^2)$$

B. Kaulakys, M. Alaburda, and V. Gontis, AIP Conf. Proc. **1129**, 13 (2009).

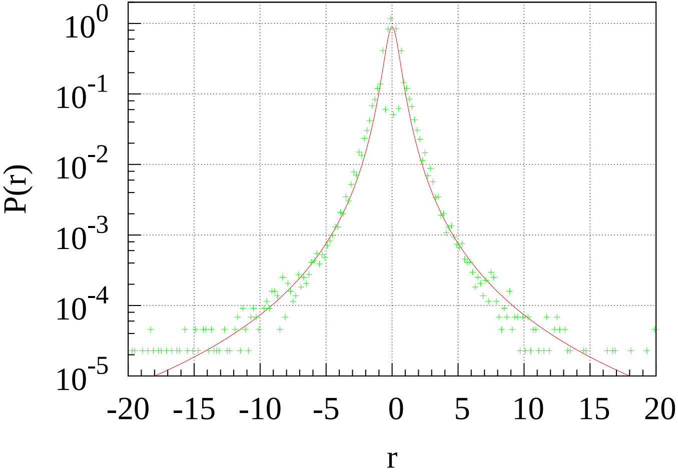
V. Gontis, B. Kaulakys, and J. Ruseckas, AIP Conf. Proc. **1129**, 563 (2009).

V. Gontis, J. Ruseckas, and A. Kononovičius, Physica A, **389**, 100 (2010).

q-exponential function: $\exp_q(x) \equiv (1 + (1 - q)x)^{1/(1-q)}$

Trading return

The distribution of normalized return r per 1 min is close to q-Gaussian.



Normalized trading return per 1 min for ABT stock on NYSE

Connection with other equations

For some choces of parameters our SDE takes the form of well-known SDE's considered in econopysics and finance.

- $\eta = 0$ and $\sigma = 1$ corresponds to the Bessel process

$$\mathrm{d}x = \frac{\delta - 1}{2} \frac{1}{x} \mathrm{d}t + \mathrm{d}W_t$$

  of dimension $\delta = 1 - \nu$

- $\eta = 1/2$, $\sigma = 2$ corresponds to the squared Bessel process

$$\mathrm{d}x = \delta \mathrm{d}t + 2\sqrt{x}\,\mathrm{d}W_t$$

  of dimension $\delta = 2(1 - \nu)$

Connection with other equations

- SDE with exponential restriction with $\eta = 1/2$, $x_{\min} = 0$ and $m = 1$ gives Cox-Ingersoll-Ross (CIR) process

$$\mathrm{d}x = k(\theta - x)\mathrm{d}t + \sigma\sqrt{x}\,\mathrm{d}W_t$$

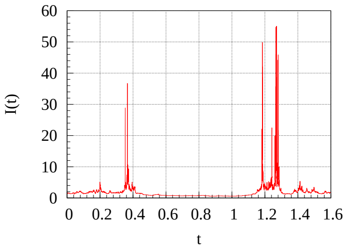
  where $k = \sigma^2/2x_{\max}$, $\theta = x_{\max}(1 - \nu)$

- When $\nu = 2\eta$, $x_{\max} = \infty$ and $m = 2\eta - 2$ then we get the Constant Elasticity of Variance (CEV) process

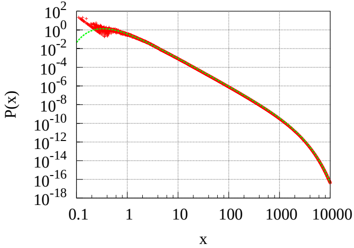
$$\mathrm{d}x = \mu x\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W_t$$

  where $\mu = \sigma^2(\eta - 1)x_{\min}^{2(\eta-1)}$

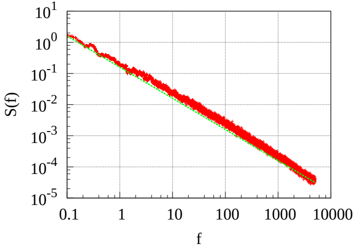
Numerical simulation



Typical signal



Distribution of x



Power spectral density

Used parameters: $\nu = 3$, $\eta = 5/2$, $x_{\mathrm{min}} = 1.0$, $x_{\mathrm{max}} = 10^3$.
$1/f$ spectrum.

Analytically solvable case

A CEV process:

$$\mathrm{d}x = \mu x \mathrm{d}t + \sigma x^{\frac{3}{2}} \mathrm{d}W_t$$

where $\mu = \sigma^2 x_{\min}/2$, $\eta = 3/2$, $\nu = 3$ and $x_{\max} = \infty$
Transition probability is

$$P_x(x', t|x, 0) = \frac{x_{\min}}{(1 - e^{-\mu t})} \sqrt{\frac{x}{x'^5}} \exp\left(\frac{1}{2}\mu t - \frac{x_{\min}}{(1 - e^{-\mu t})}\left(\frac{1}{x'} + \frac{1}{x} e^{-\mu t}\right)\right) \times I_1\left(\frac{x_{\min}}{\sinh\left(\frac{1}{2}\mu t\right)} \frac{1}{\sqrt{x x'}}\right)$$

The steady-state probability distribution has the form

$$P_0(x) = x_{\min}^2 x^{-3} \exp(-x_{\min}/x)$$

Analytically solvable case

The autocorrelation function

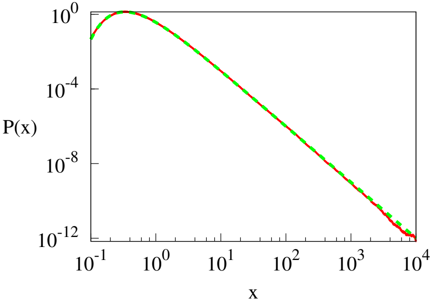
$$C(t) = -x_{\mathrm{min}}^2 e^{\mu t} \ln\left(1 - e^{-\mu t}\right)$$

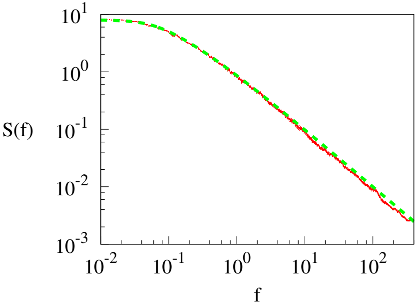
When $\mu t \ll 1$ we get $C(t) \approx -x_{\mathrm{min}}^2 \ln(\mu t)$
The power spectral density

$$S(f) = -4x_{\mathrm{min}}^2 \,\mathrm{Re}\left[\frac{\gamma + \psi(\mathrm{i}\omega/\mu)}{\mu - \mathrm{i}\omega}\right]$$

where γ is the Euler's constant and $\psi(\cdot)$ is the digamma function.
When $\omega \gg \mu$ then the power spectral density is $S(f) \approx x_{\mathrm{min}}^2/f$

Analytically solvable case





Probability distribution function $P_0(x)$ and power spectral density $S(f)$

$1/f$ noise and eigenvalues of the F-P equation

- Solutions of the Fokker-Planck equation having the form $P(x,t) = P_\lambda(x)\mathrm{e}^{-\lambda t}$ determine eigenfunctions $P_\lambda(x)$ and eigenvalues λ
- The power spectral density

$$S(f) = 4\sum_\lambda \frac{\lambda}{\lambda^2+\omega^2} X_\lambda^2\,, \qquad X_\lambda = \int_{x_{\min}}^{x_{\max}} x P_\lambda(x)\,\mathrm{d}x$$

- The shape of the power spectral density depends on the behavior of the eigenfunctions and the eigenvalues
- Expression for the power spectral density resembles the models of $1/f$ noise using the sum of the Lorentzian spectra. The Lorentzians can arise from the single nonlinear stochastic differential equation

$1/f$ noise and eigenvalues of the F-P equation

$$S(f) \approx 4 \int \frac{\lambda}{\lambda^2 + \omega^2} X_\lambda^2 D(\lambda)\,\mathrm{d}\lambda \sim \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{1}{\lambda^{\beta-1}} \frac{1}{\lambda^2 + \omega^2}\,\mathrm{d}\lambda$$

The largest contribution make the terms corresponding to the eigenvalues λ obeying the condition $\lambda_{\min} \ll \lambda \ll \lambda_{\max}$, where

$$\lambda_{\min} = \sigma^2 x_{\min}^{2(\eta-1)}\,, \qquad \lambda_{\max} = \sigma^2 x_{\max}^{2(\eta-1)}\,, \qquad \eta > 1$$
$$\lambda_{\min} = \sigma^2 x_{\max}^{2(\eta-1)}\,, \qquad \lambda_{\max} = \sigma^2 x_{\min}^{2(\eta-1)}\,, \qquad \eta < 1$$

When $\lambda_{\min} \ll \omega \ll \lambda_{\max}$ then the leading term in the expansion in the power series of ω is

$$S(f) \sim \omega^{-\beta}\,, \qquad \beta < 2$$

$1/f$ noise and eigenvalues of the F-P equation

- The shape of the power spectral density depends on the behavior of the eigenfunctions and the eigenvalues in terms of the function $X_\lambda^2 D(\lambda)$.
- One obtains $1/f^\beta$ behavior of the power spectral density when function $X_\lambda^2 D(\lambda)$ is proportional to $\lambda^{-\beta}$ for a wide range of eigenvalues λ

J. Ruseckas and B. Kaulakys, Phys. Rev. E **81**, 031105 (2010).

Summary

- We obtain a class of nonlinear SDEs, giving the power-law behavior of the power spectral density in any desirably wide range of frequencies
- and power-law steady state distribution of the signal intensity.
- The equations, as special cases, contain the well-known SDEs in economics and finance.
- One of the reasons for the appearance of the $1/f$ spectrum is the scaling property of the SDE.
- The power spectral density may be represented as a sum of the Lorentzian spectra.

Thank you for your attention!