

EXTINCTION TIMES FOR BIRTH-DEATH PROCESSES: EXACT RESULTS, CONTINUUM ASYMPTOTICS, AND THE FAILURE OF THE FOKKER–PLANCK APPROXIMATION*

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Abstract. We consider extinction times for a class of birth-death processes commonly found in applications, where there is a control parameter which defines a threshold. Below the threshold, the population quickly becomes extinct; above, it persists for a long time. We give an exact expression for the mean time to extinction in the discrete case and its asymptotic expansion for large values of the population scale. We have results below the threshold, at the threshold, and above the threshold, and we observe that the Fokker–Planck approximation is valid only quite near the threshold. We compare our asymptotic results to exact numerical evaluations for the susceptible-infected-susceptible epidemic model, which is in the class that we treat. This is an interesting example of the delicate relationship between discrete and continuum treatments of the same problem.

Key words. birth-death processes, mean first passage times, diffusion approximation

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1. Introduction. Birth-death processes are widely used as a description of phenomena in physics, chemistry [1, 2], population biology [3], and many other areas. These are Markov processes on states which we label by $n = 0, 1, \dots, R$, where R denotes the largest value allowed (which could be ∞). They are defined by the birth and death rates:

$$(1) \quad \begin{aligned} \lambda_n, & \quad n \rightarrow n + 1, \\ \mu_n, & \quad n \rightarrow n - 1. \end{aligned}$$

The processes we will consider have an absorbing state (“extinction”), which we set to $n = 0$. That is, $\lambda_0 = 0$. In what follows we will be mainly concerned with the mean time to extinction, τ_k , i.e., the mean first passage time to the state $n = 0$ starting at $n = k$. For any such Markov process with an absorbing state, extinction will occur as $t \rightarrow \infty$ with unit probability. For example, if n denotes the size of a population of organisms, we seek the mean time to biological extinction.

In this paper we will give exact expressions for the extinction time for a class of birth-death processes, and asymptotic expressions for cases where the typical n is large, i.e., in a “continuum” limit. We will investigate the validity of a popular approximation, the Fokker–Planck or diffusion method [1]. We will see that the Fokker–Planck method gives the correct asymptotic continuum behavior of τ only in very special circumstances.

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As a basis for the subsequent discussion, consider the following two processes taken from the literature of epidemiology and population biology.

- The susceptible-infected-susceptible (SIS) model of epidemiology [4]. Imagine a population of size N within which n individuals suffer from an infection and the rest, $N - n$, are susceptible. Suppose that the infection rate per contact is Λ/N , the number of contacts is $n(N - n)$, and the recovery rate is unity (fixing the unit of time). A recovered individual immediately becomes susceptible. Then

$$(2) \quad \begin{aligned} \lambda_n &= \Lambda n \left(1 - \frac{n}{N}\right), \\ \mu_n &= n. \end{aligned}$$

At the deterministic (nonstochastic, continuum) level there may then be a nonzero steady state number, n_e , of infected individuals, the solution of $\lambda_n = \mu_n$. In this SIS model $n_e = N(1 - 1/\Lambda)$, provided that $\Lambda > 1$. This model has a threshold, $\Lambda = 1$, above which the infection persists in the continuum approximation. When the $\Lambda \leq 1$, the infection dies out. In the stochastic model, however, above threshold the number of infected individuals remains near n_e for a long time (the quasi-stationary state) before the infection eventually goes extinct [5, 6].

- A logistic model from ecology [7], often called the Verhulst model. This population dynamics model assumes a birth rate per individual, B , and unit death rate per individual at low populations. In order account for competition for resources, the death rate is assumed to increase proportional to n^2 . We write

$$(3) \quad \begin{aligned} \lambda_n &= Bn, \\ \mu_n &= n + \frac{Bn^2}{N}. \end{aligned}$$

At the deterministic level, $n_e = (B - 1)N/B$, provided $B > 1$. In the continuum, for $B > 1$ the population stabilizes at n_e , while for $B \leq 1$ it goes extinct. In the stochastic model there is a quasi-stationary state for $B > 1$ in which the population fluctuates near n_e before eventually going extinct.

These two examples, both of whose continuum dynamics is the simple logistic (Verhulst) differential equation, are representative of the class of models that we consider in this paper. In these models there is a large number, N , and we assume that both λ and μ involve such a number in a special way:

$$(4) \quad \begin{aligned} \lambda_n &= N\bar{\lambda}(x), \\ \mu_n &= N\bar{\mu}(x), \end{aligned}$$

where $x = n/N$ and $\bar{\lambda}$, $\bar{\mu}$ are smooth functions of x .

These processes have the following properties: First, we assume that $\lambda_n = N\bar{\lambda}(x)$ is concave downward (or linear) and $\mu_n = N\bar{\mu}(x)$ concave upward (or linear). (We will not consider the general degenerate case in which both functions are linear.) Both functions are taken to have finite nonzero slopes near $n = 0$. The processes are most interesting when there is a control parameter so that there can be an intersection of the two curves (supersubthreshold) or not (subthreshold), depending on the parameter. We are also interested in the case when the parameter is very near threshold, in a sense that we will define below; see Figure 1.

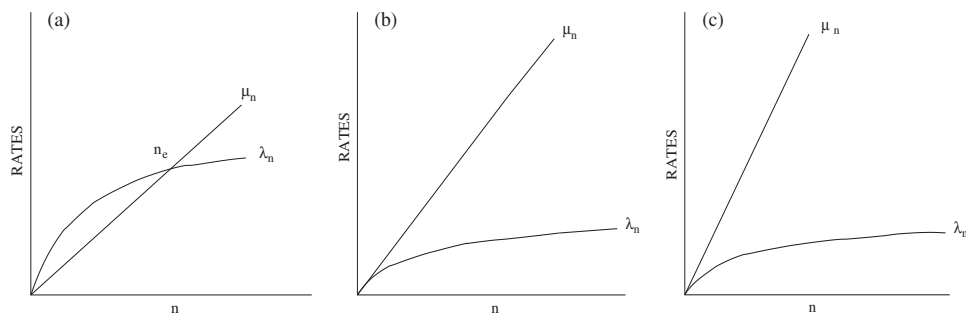


FIG. 1. Regimes for the rates λ_n, μ_n : (a) above threshold, (b) near threshold, (c) below threshold.

We are interested in the mean time to extinction starting at state n . The probabilities, $\pi_n(t)$, for the various states obey the master equation

$$(5) \quad \frac{d\pi_n(t)}{dt} = \lambda_{n-1}\pi_{n-1}(t) - (\lambda_n + \mu_n)\pi_n(t) + \mu_{n+1}\pi_{n+1}(t).$$

It is an elementary exercise [1, 2] to show that τ_n obeys

$$(6) \quad -1 = \lambda_n\tau_{n+1} - (\lambda_n + \mu_n)\tau_n + \mu_n\tau_{n-1}.$$

We will solve this equation exactly and give expressions for the asymptotic behavior of τ_k as $N \rightarrow \infty$. In the context of the SIS model, there is a very large literature on this question [8, 9, 6, 10]. Our general expression agrees with the main results of [10] for this case. However, in some cases we find different results; see below.

For $N \gg 1$, near the continuum limit, it is tempting to work directly with (6), and to try to replace it by a differential equation in x . A naive Taylor expansion gives

$$(7) \quad -1 = f(x)T'(x) + \frac{1}{2N}g^2(x)T''(x),$$

where $x = n/N$ is the continuum variable, $T(n/N) = \tau_n$, and we define

$$(8) \quad \begin{aligned} f(x) &= \bar{\lambda}(x) - \bar{\mu}(x), \\ g^2(x) &= \bar{\lambda}(x) + \bar{\mu}(x). \end{aligned}$$

The smooth functions $f(x)$ and $g^2(x)/2N$ are, respectively, the drift (sometimes called the “force”) and the diffusion coefficient.

The operator on the right-hand side of the equation for $T(x)$ above is the adjoint of the operator in the Fokker–Planck equation (FPE) [1, 2],

$$(9) \quad \partial_t P(x, t) = \left(\frac{1}{2N} \right) \partial_{xx}(g^2 P) - \partial_x(fP),$$

which is the apparent “near-continuum” limit of (5). It can be viewed as the evolution equation for the probability transition density of a random walker with diffusion coefficient $g^2/2N$ subject to a drift velocity f . This approach, which is particularly popular in the natural sciences, is based on the observation that we can think of the birth-death process as a biased random walk in n . From the work of Einstein and Schmoluchowski (see [1, 2]), we can describe a continuum random walk with a

diffusion equation, namely (9). The extinction time is the first passage time to the origin of the random walker. This approximation is attractive because the ordinary differential equation in (7) is very easy to solve, yielding a tractable formula for T in the limit of large N . Grasman and collaborators used this approach for the logistic model [7, 11] and apparently verified their results by numerical calculations. It is also possible to interpret the FPE in terms of a stochastic differential equation of the Langevin type [2], which has attractive properties.

However, as we will see below, this series of manipulations gives the correct asymptotic behavior of τ only *under very special circumstances*, namely when the two terms in (7) are comparable in size. In our case, this means that $f = \lambda - \mu$ must be very small. There are several ways in which this can happen. For example, if we are interested in small fluctuations near $x_e \equiv n_e/N$, then we can almost certainly use a Gaussian approximation and get reliable results (though we do not verify this explicitly here). This is what van Kampen [2] calls the “system size expansion,” and it is a standard method in applications.

For the extinction problem we expect that the important n 's will be all those between $n = n_e$ and $n = 0$. For the force to be small for the whole range we must be near threshold, as in Figure 1(b). As we will see below, in order for (7) to correctly describe the asymptotic behavior, we need $\rho_n = \lambda_n/\mu_n = 1 + \mathcal{O}(1/N^{1/3+\epsilon})$, $\epsilon > 0$, or, equivalently, $x_e = \mathcal{O}(N^{-1/3-\epsilon})$. We will discuss the work in [7] below and show that the case they treated was, in fact, sufficiently near threshold. This accounts for the agreement between their FPE results and numerical calculations.

This result is counterintuitive and demonstrates the delicacy of the continuum limit even for the very simple class of processes that we consider here. Explicitly, the continuum FPE is useful not for large populations, as one might guess, but only near threshold, where the quasi-equilibrium population is *much smaller* than N . The upshot is that a diffusion treatment is valid only when the drift is small. In fact, we will go further below: we will show that no diffusion description is possible for these processes (except just above threshold) in the sense that no diffusion equation can simultaneously give τ correctly and also give a good approximation to the quasi-steady state.

In section 2 we will state our results. In section 3 we give some numerical illustrations for the SIS process. In section 4 we will discuss the implications of our findings for applications. We will relegate the actual computations to the appendices.

2. Results. In this section we summarize our main results. First we briefly discuss the exact solution for the mean extinction time as a function of the initial n . Subsequently we present the large N asymptotic expansions of the solutions in the superthreshold, threshold, and subthreshold cases.

2.1. Extinction time. The first problem is to solve the second order difference equation (6) for $1 \leq n \leq R-1$, with absorption at site $n = 0$ and a reflecting boundary condition ($\lambda_R = 0$) at site $n = R$, i.e.,

$$(10) \quad \tau_0 = 0, \quad \tau_R - \tau_{R-1} = \frac{1}{\mu_R}.$$

This is straightforward [12]: let $\alpha_k = \tau_k - \tau_{k-1}$ for $k = 1, \dots, R$ so that

$$(11) \quad -1 = \lambda_n \alpha_{n+1} - \mu_n \alpha_n$$

for $1 \leq n \leq R - 1$ with the single boundary condition $\alpha_R = \frac{1}{\mu_R}$. Then the solution of this first order difference equation for α_m , $1 \leq m \leq R - 1$, is easily written down:

$$(12) \quad \alpha_m = \frac{1}{\mu_m} + \sum_{j=1}^{R-m} \frac{1}{\mu_{m+j}} \prod_{i=1}^j \rho_{m+i-1},$$

where

$$(13) \quad \rho_i = \frac{\lambda_i}{\mu_i}.$$

Then the mean extinction time, τ_n , is recovered from

$$(14) \quad \tau_n = \sum_{m=1}^n \alpha_m.$$

By regrouping the product in (12), we find the explicit solution with no approximation:

$$(15) \quad \tau_n = \sum_{m=1}^n \left[\frac{1}{\mu_m} + \prod_{i=1}^{m-1} \frac{1}{\rho_i} \sum_{j=m+1}^R \frac{1}{\mu_j} \prod_{k=1}^{j-1} \rho_k \right].$$

2.2. Expansion for large N . Now write $\lambda_n = N\bar{\lambda}(n/N)$ and $\mu_n = N\bar{\mu}(n/N)$ with $\bar{\lambda}(x)$ and $\bar{\mu}(x)$ uniformly smooth functions on $[0, r]$, where $r = R/N$. We can define $\rho_n = \bar{\rho}(n/N)$, where $\bar{\rho}(x)$ is a bounded, smooth, and nonnegative function on $[0, r]$. We will call the following quantity the “effective potential,”

$$(16) \quad \Phi(x) = - \int_0^x \log \bar{\rho}(\xi) d\xi,$$

to facilitate comparison to the FPE, below. Note that $\bar{\rho}(0) = \bar{\lambda}'(0)/\bar{\mu}'(0)$ is an indicator of the subthreshold ($\bar{\rho}(0) < 1$), threshold ($\bar{\rho}(0) = 1$), and superthreshold ($\bar{\rho}(0) > 1$) cases.

For large N the products in (15) can be estimated by the trapezoid rule of numerical analysis. Using the continuous variables $z = j/N$ and $y = m/N$,

$$(17) \quad \begin{aligned} \prod_{k=1}^{j-1} \rho_k &= \exp \left(\sum_{k=1}^{j-1} \log \bar{\rho} \left(\frac{k}{N} \right) \right) \\ &= \exp \left(N \int_0^z \log \bar{\rho}(w) dw - \frac{1}{2} (\log \bar{\rho}(0) + \log \bar{\rho}(z)) + \mathcal{O} \left(\frac{1}{N} \right) \right) \\ &= \frac{1}{\sqrt{\bar{\rho}(0)\bar{\rho}(z)}} \exp \left(N \int_0^z \log \bar{\rho}(w) dw \right) \times \left(1 + \mathcal{O} \left(\frac{1}{N} \right) \right) \\ &= \frac{1}{\sqrt{\bar{\rho}(0)\bar{\rho}(z)}} e^{-N\Phi(z)} \times \left(1 + \mathcal{O} \left(\frac{1}{N} \right) \right). \end{aligned}$$

Similarly,

$$(18) \quad \prod_{i=1}^{m-1} \frac{1}{\rho_i} = \sqrt{\bar{\rho}(0)\bar{\rho}(y)} e^{N\Phi(y)} \times \left(1 + \mathcal{O} \left(\frac{1}{N} \right) \right).$$

In these estimates and others to follow, the coefficients of the $\mathcal{O}(N^{-\beta})$ error terms depend on the regularity of the rate functions $\bar{\lambda}(x)$ and $\bar{\mu}(x)$. We proceed under the assumption of sufficient smoothness so that the estimates are valid.

In the following we write $x = n/N$ for the initial point. The large N asymptotic behavior of $\bar{\tau}(x) = \tau_n$, as given in (15), is very different for the superthreshold, threshold, and subthreshold cases, as follows.

- *Superthreshold case.* When $\bar{\lambda}'(0) > \bar{\mu}'(0)$, there is a unique “equilibrium” state n_e where $\lambda_{n_e} = \mu_{n_e}$, or equivalently a unique “deterministic steady state” $x_e = n_e/N > 0$ where $\bar{\lambda}(x_e) = \bar{\mu}(x_e)$. (In the SIS and logistic models this corresponds, respectively, to the conditions $\Lambda > 1$ and $B > 1$.) For the stochastic processes, the extinction time is exponentially large as $N \rightarrow \infty$. We find that $\bar{\tau}(x) \sim \tau_{n_e} = \bar{\tau}(x_e)$ for $n = \mathcal{O}(N)$, i.e., $x = \mathcal{O}(1)$. Further,

$$(19) \quad \bar{\tau}(x_e) = \sqrt{\frac{2\pi\bar{\rho}(0)}{N[\bar{\lambda}(x_e)\bar{\mu}'(x_e) - \bar{\lambda}'(x_e)\bar{\mu}(x_e)]}} \times \frac{e^{-N\Phi(x_e)}}{\bar{\rho}(0) - 1} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

In this region $\bar{\tau}(x)$ is independent of x . For $x = \mathcal{O}(\infty/N)$ there is a boundary layer where $\bar{\tau}(x)$ depends on x . However, we have found a simple correction factor which allows us to give, for all $x \in [0, r = R/N]$, a uniform asymptotic approximation,

$$(20) \quad \bar{\tau}(x) = \left(1 - e^{-N \log \bar{\rho}(0)x}\right) \bar{\tau}(x_e).$$

- *Threshold case.* Here $\lambda_n < \mu_n$ for $n \geq 1$, but the derivatives of $\bar{\lambda}(x)$ and $\bar{\mu}(x)$ at 0 are equal. (In the SIS and logistic models this corresponds, respectively, to $\Lambda = 1$ and $B = 1$.) In this critical situation the dominant term in the large N asymptotic expansion of the extinction time is $\propto N^{\frac{1}{2}}$:

$$(21) \quad \bar{\tau}(x) = \frac{(\pi/2)^{\frac{3}{2}}}{\sqrt{\Phi''(0)\bar{\lambda}'(0)\bar{\mu}'(0)}} \times \sqrt{N} + \frac{\log Nx}{\bar{\mu}'(0)} + \mathcal{O}(1)$$

for $n/N = x = \mathcal{O}(1)$.

- *Subthreshold case.* Below threshold, $\lambda_n < \mu_n$ for all $n \geq 1$, and so $\bar{\rho}(x) < 1$ for all $x \geq 0$. Moreover, the derivative of $\bar{\lambda}(x)$ is strictly smaller than the derivative of $\bar{\mu}(x)$ for all $x \geq 0$. (In the SIS and logistic models this corresponds, respectively, to $\Lambda < 1$ and $B < 1$.) The deterministic evolution tends rapidly towards extinction in this case. For the stochastic process, the extinction time is logarithmic in N as $N \rightarrow \infty$:

$$(22) \quad \bar{\tau}(x) = \frac{1}{\bar{\mu}'(0)(1 - \bar{\rho}(0))} \log Nx + \mathcal{O}(1)$$

for $n/N = x = \mathcal{O}(1)$.

3. Numerical estimates and the failure of the Fokker–Planck approximation. In this section we compare our asymptotic results to numerical calculations and to the FPE approximation for the SIS model. The numerical results here are a direct evaluation of the exact formula (15).

3.1. Numerical and analytical results for the SIS model. As we pointed out above, an example of the family of models considered here is the SIS model defined by

$$(23) \quad \lambda_n = \Lambda n \left(1 - \frac{n}{N}\right), \quad \mu_n = n.$$

For this case $0 \leq x \leq 1$ and

$$(24) \quad \bar{\lambda}(x) = \Lambda x(1 - x), \quad \bar{\mu}(x) = x,$$

so that

$$(25) \quad \bar{\rho}(x) = \Lambda(1 - x)$$

and

$$(26) \quad \Phi(x) = - \int_0^x \log \Lambda(1 - \xi) d\xi = (1 - x) \log \Lambda(1 - x) + x - \log \Lambda.$$

It is easy to check that $\Phi(x)$ is a convex function.

The analytical results for this model based on (19), (21), and (22) are presented in the following table.

TABLE 1

	Λ	$\Phi(x)$	τ_n for $n/N = x = \mathcal{O}(1)$	
Superthreshold	$\Lambda > 1$	$\Phi'(0) < 0$	$\frac{\Lambda}{(\Lambda-1)^2} \sqrt{\frac{2\pi}{N}} e^{N(\log \Lambda - 1 + 1/\Lambda)} \times (1 + \mathcal{O}(\frac{1}{N}))$	Figure 2
Threshold	$\Lambda = 1$	$\Phi'(0) = 0$	$(\frac{\pi}{2})^{\frac{3}{2}} \sqrt{N} + \log Nx + \mathcal{O}(1)$	Figure 4
Subthreshold	$\Lambda < 1$	$\Phi'(0) > 0$	$\frac{1}{1-\Lambda} \log Nx + \mathcal{O}(1)$	Figure 5

The superthreshold case is compared with numerical simulations in Figure 2. Our formula agrees with the results of [10] and with the exact results.

We can go further and test the validity of our error estimate in the first line of the table above, and also our treatment of the boundary layer. We define the relative error as $\bar{\tau}/\bar{\tau}_{asy} - 1$, where $\bar{\tau}_{asy}$ is given in (19) and (20). We plot N times this quantity in Figure 3 to show that the relative error is of order $1/N$ uniformly in x .

The threshold case is shown in Figure 4; we find good agreement between our asymptotic formula and the exact results. For the subthreshold case our formulas do not agree with [10], but they do agree with the numerical results. This is shown in Figure 5.

3.2. Fokker–Planck approximation. The Fokker–Planck approach approximates the finite difference equation (6) by the differential equation (7), solves it, and then extracts the large N asymptotics. This has the advantage of producing a tractable and generally useful partial differential equation, (9). However, as we will see, it does not give the correct answers in general for our class of processes. In this section we will be concerned only with the superthreshold case.

3.2.1. Asymptotic estimates. The solution to (7) is (neglecting $\mathcal{O}(1/N^2)$)

$$(27) \quad T(x) = \frac{1}{\bar{\mu}(r)} \int_0^x e^{-N(V(r)-V(y))} dy + 2N \int_0^x \int_y^r \frac{e^{-N(V(z)-V(y))}}{\bar{\lambda}(z) + \bar{\mu}(z)} dz dy,$$

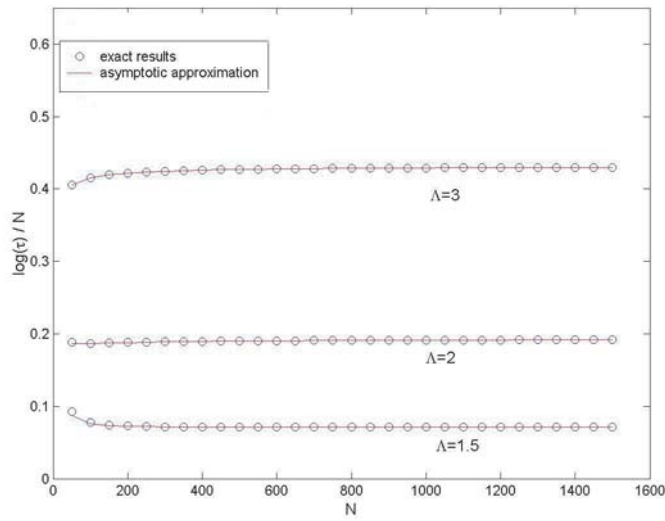


FIG. 2. Comparison of analytical and numerical results for the SIS model in the superthreshold case. $\log \bar{\tau}(x_e)/N$ is plotted as a function of N for three values of Λ .

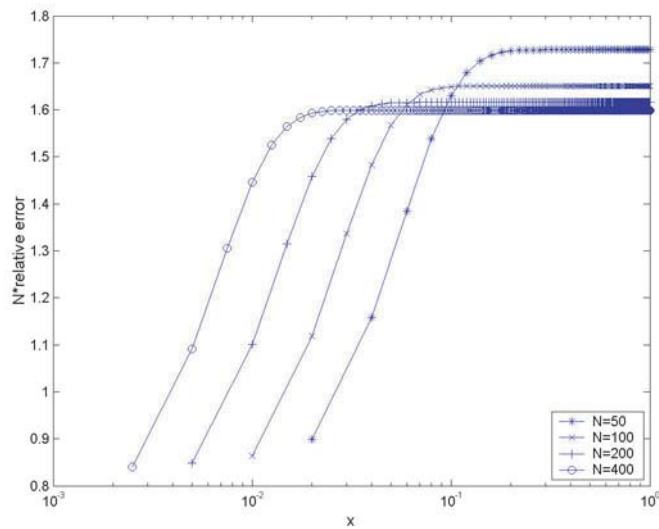


FIG. 3. N times the relative error for the superthreshold case of the SIS model as a function of x for various N and $\Lambda = 3$.

where

$$(28) \quad V(x) = -2 \int_0^x \frac{\bar{\lambda}(\xi) - \bar{\mu}(\xi)}{\bar{\lambda}(\xi) + \bar{\mu}(\xi)} d\xi \equiv -2 \int_0^x \frac{f(\xi)}{g^2(\xi)} d\xi$$

plays the role of the effective potential.

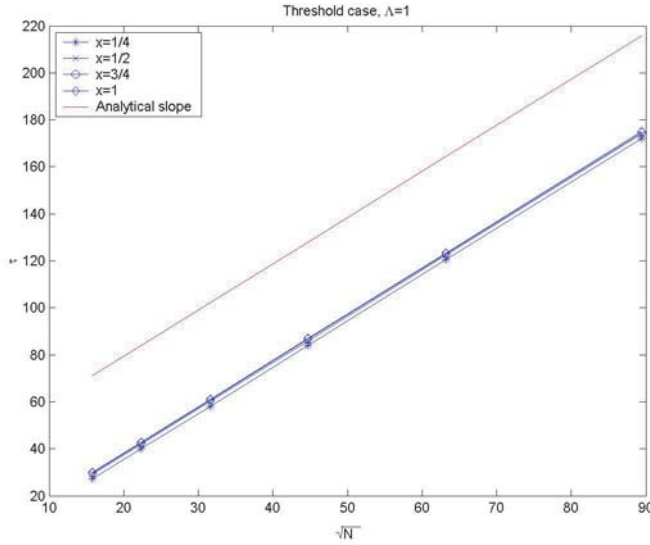


FIG. 4. Comparison of analytical and numerical results for the SIS model in the threshold case. $\bar{\tau}(x)$ is plotted as a function of $N^{1/2}$ for various values of x . The symbols are exact results of (15), and the dashed lines are guides for the eye. The solid line shows the prediction for the slope in (21). The vertical position of the solid line is arbitrary.

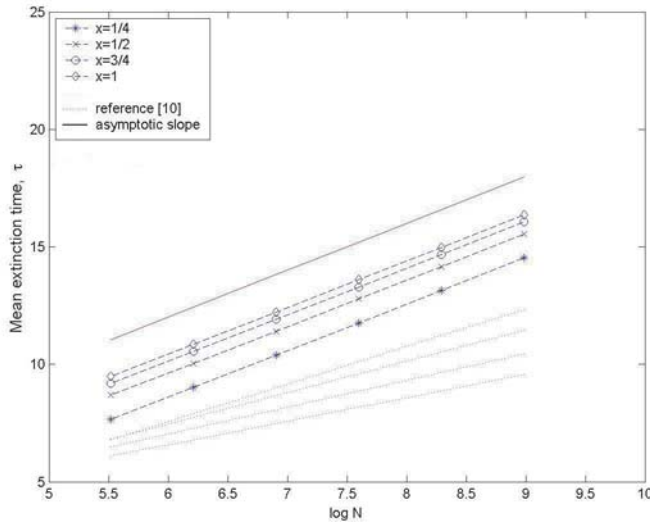


FIG. 5. Comparison of analytical and numerical results for $\bar{\tau}(x)$ in the subthreshold case of the SIS model for $\Lambda = 1/2$. $\bar{\tau}(x)$ is plotted as a function of $\log N$ for several values of x . The symbols are exact results, and the dashed lines are guides for the eye. The solid line shows the prediction for the slope from (22). The vertical position of the solid line is arbitrary. The results of [10] are also shown as dotted lines which correspond to $x = 1/4, 1/2, 3/4, 1$ (from top to bottom).

The asymptotic behavior, using standard techniques [1], is

$$(29) \quad T(x) \approx T(x_e) = \frac{2}{|V'(0)|} \sqrt{\frac{2\pi}{NV''(x_e)}} \frac{e^{-NV(x_e)}}{g^2(x_e)} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

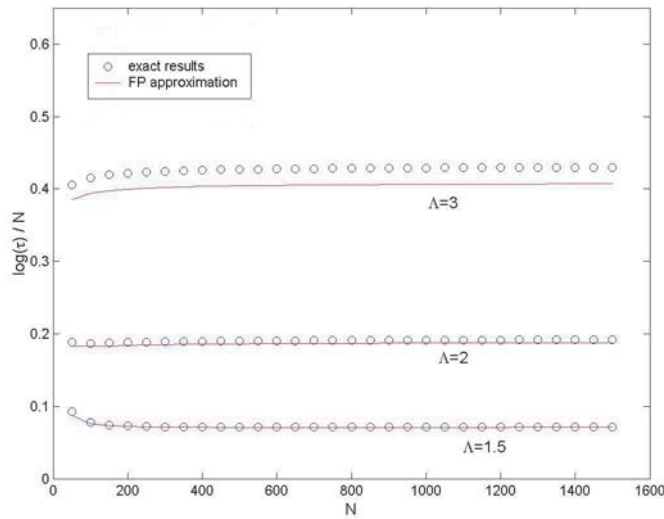


FIG. 6. Comparison of the FPE and numerical results for the SIS model in the superthreshold case. $\log \bar{\tau}(x_e)/N$ is plotted as a function of N for several values of Λ .

For the special case of the SIS model we find

$$(30) \quad T(x) \approx \frac{\Lambda + 1}{2\sqrt{\Lambda}} \frac{\Lambda}{(\Lambda - 1)^2} \sqrt{\frac{2\pi}{N}} e^{-NV(x_e)} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right) \right),$$

where

$$(31) \quad -V(x_e) = \frac{4}{\Lambda} \log \frac{2}{\Lambda + 1} + 2 \frac{\Lambda - 1}{\Lambda}.$$

This expression is quite different from the first line of Table 1 (which agrees with the exact results). In Figure 6 we show a comparison of (30) with the exact results. It is clear that there is a significant discrepancy for large values of Λ , that is, far from the threshold at $\Lambda = 1$. Note that $\log \bar{\tau}(x_e)/N$ is plotted in the figure. From the figure we see that, for $N = 1000$, T is a factor of about 10^7 smaller than the exact result.

We have done a similar calculation for the logistic model. The numerical results of [7] are all near to threshold, so that the apparent numerical verification of their FPE calculation is of only limited validity. Far above threshold there are similar very large discrepancies between T and $\bar{\tau}$.

3.2.2. The effective potentials. We now turn to the source of the problem with the FPE estimate of $\bar{\tau}$. For the superthreshold case, $\bar{\tau}$ in (29) is of the standard form of a relatively slowly varying prefactor multiplied by $\exp(-NV)$, where V is given in (28). In the following discussion we will ignore the prefactor and consider only the dominant exponential term. The FPE gives a V which is not the same as the correct answer, Φ , from (16).

However, V is close to Φ quite near the threshold, in which case the force is small over the important range of x , namely $[0, x_e]$. To see this, set $\bar{\lambda}(x) - \bar{\mu}(x) = f_N(x)$. Assume that the force is, in fact, small: $f_N(x) \rightarrow 0$ appropriately uniformly in x as $N \rightarrow \infty$. Then

$$(32) \quad \frac{d\Phi(x)}{dx} = -\frac{f_N(x)}{\bar{\mu}(x)} + \frac{1}{2} \left(\frac{f_N(x)}{\bar{\mu}(x)} \right)^2 - \frac{1}{3} \left(\frac{f_N(x)}{\bar{\mu}(x)} \right)^3 + \dots,$$

while

$$(33) \quad \frac{dV(x)}{dx} = -\frac{f_N(x)}{\bar{\mu}(x)} + \frac{1}{2} \left(\frac{f_N(x)}{\bar{\mu}(x)} \right)^2 - \frac{1}{4} \left(\frac{f_N(x)}{\bar{\mu}(x)} \right)^3 + \dots.$$

Hence the exponential terms formally agree to a factor of $(1 + \mathcal{O}(1/N))$ only if $f_N(x) = \bar{\lambda}(x) - \bar{\mu}(x) \leq \mathcal{O}(N^{-1/3})$; that is, the drift must be small over the whole range of x . Equivalently, we can write $\rho_n = \lambda_n/\mu_n = 1 + \mathcal{O}(1/N^{1/3+\epsilon})$, $\epsilon > 0$, or $x_e = \mathcal{O}(N^{-1/3-\epsilon})$. If this is true then the two estimates of $\bar{\tau}$ differ by factors of order unity.

3.2.3. “Corrected” FPE. We might be tempted to define a corrected FPE, with an effective potential Φ , by redefining f and g^2 in (9). A suitable f and g^2 could be found to do this, but the corrected FPE would not be particularly useful in general. The point of using (9) is to have a unified description of the process, equivalent in the continuum limit, for the original master equation. In particular, we should be able to describe the quasi-stationary distribution with the same equation. We will show that this is not possible.

A version of the quasi-stationary distribution [6] is obtained by changing the boundary condition at the origin to reflecting; that is, we set $\lambda_0 = 1$. Then a stationary distribution exists. We can find this by returning to (5), setting $d\pi_n(t) = 0$, and solving the equation. The result is

$$(34) \quad \pi_n = \frac{\prod_{j=0}^{n-1} [\lambda_j/\mu_{j+1}]}{1 + \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} [\lambda_j/\mu_{j+1}]}.$$

We take the continuum limit by defining $p(n/N) = N\pi_n$ and using the method of (17). We have

$$(35) \quad p(x) \propto \frac{1}{\sqrt{\lambda(x)\bar{\mu}(x)}} e^{-N\Phi(x)} \left(1 + \mathcal{O}\left(\frac{1}{N}\right) \right).$$

We can also solve for the stationary state of (9), with the requirement that the effective potential be Φ . This gives

$$(36) \quad P(x) \propto \frac{e^{-N\Phi(x)}}{g^2}.$$

Comparing these two equations, we see that we must set

$$(37) \quad \begin{aligned} g^2(x) &= K\sqrt{\lambda(x)\bar{\mu}(x)}, \\ f(x) &= 2K\sqrt{\lambda(x)\bar{\mu}(x)} \log \frac{\bar{\lambda}(x)}{\bar{\mu}(x)}, \end{aligned}$$

where K is a constant. We can also recalculate T , as in (29). If we compare the result to (19), we see that, while we have the correct effective potential (by construction), we do not get the correct prefactor, so that T is inconsistent with $\bar{\tau}$.

4. Summary and discussion. We have described two sorts of results in this paper. On the one hand, we have shown how to generalize previous work on the SIS model [8, 9, 6, 10] to a large class of birth-death processes as well as to the threshold and subthreshold cases. We have extended and improved the mean extinction time results of [10] for the threshold and subthreshold cases.

Our treatment of the FPE should be viewed mainly as a warning about the subtlety of the relationship between discrete and continuum approaches. In cases where the continuum equation (9) *should* work, it fails if the drift is too large. This is reflected in the fact that the exact effective potential, $\Phi = -\int[\log(1 + f/g^2) - \log(1 - f/g^2)]$, is different from the FPE expression, $V = -2\int[f/g^2]$. They are the same for uniformly small f/g^2 . For practical purposes, if the process is not immediately above threshold, it is preferable to use (19), or even the exact expression, (15). The best diffusion approximation that we can propose would use the exact potential Φ , and would give only the dominant exponential in $\bar{\tau}$ but not the prefactor.

For a problem with a wide separation of scales, the option of using an exact expression is often not available. For example, in work on modeling calcium waves in cells [13], the underlying processes are too complex to allow a practical exact calculation. One method for circumventing this difficulty is to use a Langevin equation, replacing some of the rapid processes by noise terms. In practice, however, there is some evidence that such a method gives acceptable results only near an appropriately defined threshold [14] (see also [15]). It is possible that the diffusion approximation mentioned at the end of the last paragraph would be useful in this case.

We have tried to produce a heuristic argument for the quantitative failure of the FPE approximation for the mean extinction time in case the drift is not sufficiently small. We approach the problem from the other direction by asking when it is possible to accurately approximate a continuum diffusion process by a discrete state Markov process. When the state space is discretized, the difficulty is that the transition times between neighboring states (separated by distance $1/N$) are not exponentially distributed for the diffusion process, so the “imbedded” process is not Markovian. In order for the discrete state process to mimic a Markov process, the transition times must be close to exponential random variables. Although a more precise condition relating the drift and diffusion (and N) would require calculations that we do not perform here, we speculate that when the diffusion coefficient is $\mathcal{O}(1/N)$ and the state-space discretization $\mathcal{O}(1/N)$ as well, it is simply impossible to sufficiently approximate an exponential distribution for the transition time between neighboring states as $N \rightarrow \infty$ unless the drift is small enough.

Appendix. In this appendix we give the details of the large N expansion of (15), which we repeat here for convenience:

$$(38) \quad \tau_n = \sum_{m=1}^n \left[\frac{1}{\mu_m} + \overbrace{\prod_{i=1}^{m-1} \frac{1}{\rho_i} \sum_{j=m+1}^R \frac{1}{\mu_j} \prod_{k=1}^{j-1} \rho_k}^{A_m} \right].$$

In the following sections we discuss the superthreshold, threshold, and subthreshold cases. The major computations are of the term $\sum A_m$ in (38).

Appendix A.1. Superthreshold case. In this situation $\Phi(x)$ is convex with a quadratic minimum at x_e , the unique solution of $\bar{\lambda}(x_e) = \bar{\mu}(x_e)$. Thus, putting

$s = m/N$ and invoking standard integral estimates,

$$(39) \quad \sum_{j=m+1}^R \frac{1}{\mu_j} \prod_{k=1}^{j-1} \rho_k = \sum_{j=m+1}^R \frac{1}{\mu_j} \frac{1}{\sqrt{\bar{\rho}(0)\bar{\rho}(j/N)}} e^{-N\Phi(\frac{j}{N})} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\ = \int_s^r \frac{dz}{\sqrt{\bar{\rho}(0)\bar{\rho}(z)\bar{\mu}(z)}} e^{-N\Phi(z)} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

For the next step we use the fact that if h is a smooth function with $h(x_e) \neq 0$, then for $s < x_e$,

$$(40) \quad \int_s^r h(\xi) e^{-N\Phi(\xi)} d\xi = h(x_e) \sqrt{\frac{2\pi}{N\Phi''(x_e)}} e^{-N\Phi(x_e)} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

We now use $n = O(N)$, $\Phi'(0) < 0$, $\Phi(0) = 0$ and the smoothness of $\bar{\rho}$ to compute the geometric sum after expanding $\Phi(m/N)$ around 0. Thus,

$$(41) \quad \sum_{m=1}^n A_m = \sum_{m=1}^n \frac{1}{\bar{\mu}(x_e)} \sqrt{\frac{\bar{\rho}(m/N)}{\bar{\rho}(x_e)}} \sqrt{\frac{2\pi}{N\Phi''(x_e)}} e^{-N\Phi(x_e)} e^{N\Phi(\frac{m}{N})} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\ = \sqrt{\frac{2\pi}{N\Phi''(x_e)}} \frac{e^{-N\Phi(x_e)}}{\bar{\mu}(x_e)\sqrt{\bar{\rho}(x_e)}} \sum_{m=1}^n \sqrt{\bar{\rho}\left(\frac{m}{N}\right)} e^{N\Phi(\frac{m}{N})} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\ = \sqrt{\frac{2\pi}{N\Phi''(x_e)}} \frac{e^{-N\Phi(x_e)}}{\bar{\mu}(x_e)\sqrt{\bar{\rho}(x_e)}} \frac{\sqrt{\bar{\rho}(0)}}{1 - e^{\Phi'(0)}} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

The last expression is exponentially large in N , and thus $\sum_{m=1}^n \frac{1}{\mu_m}$, which we estimate in the subthreshold case below, is completely negligible. Eliminating derivatives of $\Phi(x)$, we then find

$$(42) \quad \tau_n = \sqrt{\frac{2\pi\bar{\rho}(0)}{N[\bar{\lambda}(x_e)\bar{\mu}'(x_e) - \bar{\lambda}'(x_e)\bar{\mu}(x_e)]}} \frac{e^{-N\Phi(x_e)}}{\bar{\rho}(0) - 1} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right),$$

independent of n for $n = O(N)$. The boundary layer correction follows simply from a finite geometric series approximation to the sum in the penultimate line in (41).

Appendix A.2. Threshold case. Here the potential function $\Phi(x)$ is convex with vanishing derivative but nonvanishing curvature at $x = 0$. Referring to the terms in the exact solution (38), invoking the trapezoid approximation, and setting $s = m/N$,

$$(43) \quad \sum_{m=1}^n A_m = \sum_{m=1}^n \int_{m/N}^r \frac{\sqrt{\bar{\rho}(m/N)}}{\sqrt{\bar{\rho}(z)\bar{\mu}(z)}} e^{-N(\Phi(z) - \Phi(\frac{m}{N}))} dz \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\ = N \int_0^x ds \int_s^r dz \frac{\sqrt{\bar{\rho}(s)}}{\sqrt{\bar{\lambda}(z)\bar{\mu}(z)}} e^{-N(\Phi(z) - \Phi(s))} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

Most of the contribution to the integral comes from a neighborhood of the origin, so we proceed making standard integral approximations. Expand $\Phi(x)$, $\bar{\rho}(s)$, $\bar{\lambda}(z)$, and $\bar{\mu}(z)$ using $\bar{\lambda}(0) = \bar{\mu}(0) = \Phi(0) = \Phi'(0) = 0$:

$$(44) \quad \sum_{m=1}^n A_m = N \frac{\sqrt{\bar{\rho}(0)}}{\sqrt{\bar{\lambda}'(0)\bar{\mu}'(0)}} \int_0^x ds \int_s^r dz \frac{e^{-N\Phi''(0)(\frac{z^2}{2} - \frac{s^2}{2})}}{z} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

Switching the integrals,

$$(45) \quad \sum_{m=1}^n A_m = N \frac{\sqrt{\bar{\rho}(0)}}{\sqrt{\lambda'(0)\bar{\mu}'(0)}} \int_0^r \frac{e^{-N\Phi''(0)\frac{z^2}{2}}}{z} dz \int_0^{\min(z,x)} e^{N\Phi''(0)\frac{s^2}{2}} ds \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right),$$

and changing variables,

$$(46) \quad = \sqrt{\frac{N}{\Phi''(0)}} \frac{\sqrt{\bar{\rho}(0)}}{\sqrt{\lambda'(0)\bar{\mu}'(0)}} \int_0^{r\sqrt{N\Phi''(0)}} \frac{e^{-\frac{v^2}{2}}}{v} dv \int_0^{\min(v,\sqrt{N\Phi''(0)x})} e^{\frac{u^2}{2}} du \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

Then because N is large we may modify the limits of the integrals:

$$(47) \quad \sum_{m=1}^n A_m = \sqrt{\frac{N}{\Phi''(0)}} \frac{\sqrt{\bar{\rho}(0)}}{\sqrt{\lambda'(0)\bar{\mu}'(0)}} \int_0^\infty dv \frac{e^{-\frac{v^2}{2}}}{v} \int_0^v du e^{\frac{u^2}{2}} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right).$$

Integrate $e^{u^2/2}$ using its Taylor expansion and use the formula for the even moments of a Gaussian to obtain

$$(48) \quad \begin{aligned} \sum_{m=1}^n A_m &= \sqrt{\frac{N}{\Phi''(0)}} \frac{\sqrt{\bar{\rho}(0)}}{\sqrt{\lambda'(0)\bar{\mu}'(0)}} \sum_{k=0}^\infty \frac{1}{2^k k! (2k+1)} \int_0^\infty v^{2k} e^{-\frac{v^2}{2}} dv \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\ &= \sqrt{\frac{N}{\Phi''(0)}} \frac{\sqrt{\bar{\rho}(0)}}{\sqrt{\lambda'(0)\bar{\mu}'(0)}} \sum_{k=0}^\infty \frac{(2k)!}{(2^k k!)^2 (2k+1)} \sqrt{\frac{\pi}{2}} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \end{aligned}$$

The sum above converges (slowly) to $\frac{\pi}{2}$, and thus

$$(49) \quad \sum_{m=1}^n A_m = \frac{\pi}{2} \sqrt{\frac{\pi}{2\Phi''(0)}} \sqrt{\frac{\bar{\rho}(0)}{\lambda'(0)\bar{\mu}'(0)}} \sqrt{N} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

We show in the next section that

$$(50) \quad \sum_{m=1}^n \frac{1}{\mu_m} = \frac{1}{\bar{\mu}'(0)} \log n + \mathcal{O}(1).$$

Thus, recalling $\bar{\rho}(0) = 1$ in this case,

$$(51) \quad \tau_n = \frac{\left(\frac{\pi}{2}\right)^{\frac{3}{2}}}{\sqrt{\Phi''(0)\lambda'(0)\bar{\mu}'(0)}} \sqrt{N} + \frac{1}{\bar{\mu}'(0)} \log n + \mathcal{O}(1).$$

Appendix A.3. Subthreshold case. Here the potential function $\Phi(x)$ is a convex function with positive derivative at 0. From (38),

$$(52) \quad \begin{aligned} A_m &= \sqrt{\bar{\rho}\left(\frac{m}{N}\right)} \sum_{j=m+1}^R \frac{1}{N\bar{\mu}(j/N)\sqrt{\bar{\rho}(j/N)}} e^{N(\Phi(\frac{m}{N})-\Phi(\frac{j}{N}))} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\ &= \sqrt{\bar{\rho}\left(\frac{m}{N}\right)} \sum_{j=m+1}^R \frac{\bar{\rho}(m/N)^{j-m}}{N\sqrt{\lambda(j/N)\bar{\mu}(j/N)}} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\ &= \bar{\rho}\left(\frac{m}{N}\right)^{-m+\frac{1}{2}} \sum_{j=m+1}^R \frac{\bar{\rho}(m/N)^j}{N\sqrt{\lambda(j/N)\bar{\mu}(j/N)}} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \end{aligned}$$

Adding and subtracting

$$(53) \quad \sum_{j=m+1}^R \frac{\bar{\rho}(m/N)^{j-m+\frac{1}{2}}}{N\sqrt{\lambda(m/N)\bar{\mu}(m/N)}}$$

and recalling that in the subthreshold case $\bar{\rho}(x) < 1$ uniformly for $x \in [0, r]$, we have

$$(54) \quad \begin{aligned} A_m &= \bar{\rho} \left(\frac{m}{N}\right)^{-m+\frac{1}{2}} \sum_{j=m+1}^R \left(\frac{\bar{\rho}(m/N)^j}{N\sqrt{\lambda(j/N)\bar{\mu}(j/N)}} - \frac{\bar{\rho}(m/N)^j}{N\sqrt{\lambda(m/N)\bar{\mu}(m/N)}} \right) \\ &\quad + \sum_{j=m+1}^R \frac{\bar{\rho}(m/N)^{j-m+\frac{1}{2}}}{N\sqrt{\lambda(m/N)\bar{\mu}(m/N)}} \\ &= \bar{\rho} \left(\frac{m}{N}\right)^{-m+\frac{1}{2}} \underbrace{\int_s^r \left(\frac{\bar{\rho}(s)^{N\xi}}{\sqrt{\lambda(\xi)\bar{\mu}(\xi)}} - \frac{\bar{\rho}(s)^{N\xi}}{\sqrt{\lambda(s)\bar{\mu}(s)}} \right) d\xi}_{B_m} \\ &\quad + \underbrace{\frac{\bar{\rho}(m/N)}{1-\bar{\rho}(m/N)} \frac{1}{N\bar{\mu}(m/N)}}_{C_m} + \mathcal{O}\left(\frac{1}{N}\right) \\ &= B_m + C_m + \mathcal{O}\left(\frac{1}{N}\right). \end{aligned}$$

Now we evaluate $\sum_{m=1}^n B_m$ and $\sum_{m=1}^n C_m$ separately as $n = Nx \rightarrow \infty$. Changing variables in the integral in B_m and using the facts that, for small s , $\sqrt{\lambda(s)\bar{\mu}(s)} = \sqrt{\lambda(0)\bar{\mu}(0)} s + \mathcal{O}(s^2)$, or $\sqrt{\lambda(s)\bar{\mu}(s)} = \sqrt{\lambda(0)\bar{\mu}(0)} s \times (1 + \mathcal{O}(s))$, and $\bar{\rho}(s) = \mathcal{O}(1)$, we find

$$(55) \quad \begin{aligned} B_m &= \int_0^{r-s} \left(\frac{\bar{\rho}(s)^{N\xi+\frac{1}{2}}}{\sqrt{\lambda(\xi+s)\bar{\mu}(\xi+s)}} - \frac{\bar{\rho}(s)^{N\xi+\frac{1}{2}}}{\sqrt{\lambda(s)\bar{\mu}(s)}} \right) d\xi \\ &= \mathcal{O}(1) \times \int_0^{r-s} \bar{\rho}(s)^{N\xi+\frac{1}{2}} \frac{\xi}{s(\xi+s)} d\xi \\ &= \mathcal{O}(1) \times \int_0^{r-s} e^{N\xi \log \bar{\rho}(s)} \frac{\xi}{s(\xi+s)} d\xi \\ &= \mathcal{O}(1) \times \int_0^{r/s-1} e^{(s\alpha)z} \frac{z}{1+z} dz \\ &\leq \mathcal{O}(1) \times \int_0^{r/s-1} e^{(s\alpha)z} z dz \\ &\leq \mathcal{O}(1) \times \frac{1}{(s\alpha)^2}, \end{aligned}$$

where we define $\alpha = N \log \bar{\rho}(s)$. Then, recalling $s = m/N$, we conclude

$$(56) \quad \sum_{m=1}^n B_m \leq \mathcal{O}(1) \times \sum_{m=1}^n \frac{1}{m^2(\log \bar{\rho}(m/N))^2} = \mathcal{O}(1)$$

as $N \rightarrow \infty$ (which means $n \rightarrow \infty$ as well).

On the other hand, the C_m terms may be written

$$(57) \quad \sum_{m=1}^n C_m = \sum_{m=1}^n \eta\left(\frac{m}{N}\right) \frac{1}{N\bar{\mu}(m/N)},$$

where we define

$$(58) \quad \eta\left(\frac{m}{N}\right) = \frac{\bar{\rho}(m/N)}{1 - \bar{\rho}(m/N)}.$$

Then subtract and add the “divergent part” of the sum:

$$(59) \quad \begin{aligned} \sum_{m=1}^n C_m &= \sum_{m=1}^n \left[\frac{\eta(m/N)}{N\bar{\mu}(m/N)} - \frac{\eta(0)}{m\bar{\mu}'(0)} \right] + \sum_{m=1}^n \frac{\eta(0)}{m\bar{\mu}'(0)} \\ &= \underbrace{\int_0^x \frac{\eta(\xi)\xi\bar{\mu}'(0) - \eta(0)\bar{\mu}(\xi)}{\xi\bar{\mu}'(0)\bar{\mu}(\xi)} d\xi}_{\mathcal{O}(1)} + \frac{\eta(0)}{\bar{\mu}'(0)}(\gamma + \log n) + \mathcal{O}\left(\frac{1}{N}\right) \\ &= \frac{\eta(0)}{\bar{\mu}'(0)} \log(n) + \mathcal{O}(1), \end{aligned}$$

where $\gamma = 0.5772\dots$ is Euler’s constant.

Finally,

$$(60) \quad \begin{aligned} \sum_{m=1}^n \frac{1}{\mu_m} &= \sum_{m=1}^n \frac{1}{N\bar{\mu}(m/N)} \\ &= \frac{\gamma + \log(n)}{\bar{\mu}'(0)} + \underbrace{\int_0^x \frac{\xi\bar{\mu}'(0) - \bar{\mu}(\xi)}{\xi\bar{\mu}'(0)\bar{\mu}(\xi)} d\xi}_{\mathcal{O}(1)} + \mathcal{O}\left(\frac{1}{N}\right) \\ &= \frac{1}{\bar{\mu}'(0)} \log(n) + \mathcal{O}(1). \end{aligned}$$

Putting these calculations together, we conclude

$$(61) \quad \tau_n = \bar{\tau}(x) = \frac{\eta(0)}{\bar{\mu}'(0)} \log n + \frac{1}{\bar{\mu}'(0)} \log n + \mathcal{O}(1) = \frac{1}{\bar{\mu}'(0)(1 - \bar{\rho}(0))} \log Nx + \mathcal{O}(1).$$

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