3. $$E_1 = \frac{h^2}{8mL^2} = 5.6\text{ eV}$$

With $L' = 2L$,

$$E_1' = \frac{h^2}{8mL'^2} = \frac{h^2}{8m(2L)^2} = \frac{1}{4}\frac{h^2}{8mL^2} = \frac{1}{4}(5.6\text{ eV}) = 1.4\text{ eV}$$

4. With $\lambda_n = 2L/n$,

$$\lambda_1 = \frac{2(0.144\text{ nm})}{1} = 0.288\text{ nm} \qquad \lambda_2 = \frac{2(0.144\text{ nm})}{2} = 0.144\text{ nm} \qquad \lambda_3 = \frac{2(0.144\text{ nm})}{3} = 0.096\text{ nm}$$

5. The smallest energy is (using Equation 5.3)

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\text{ eV}\cdot\text{nm})^2}{8(511{,}000\text{ eV})(0.062\text{ nm})^2} = 98\text{ eV}$$

Then $E_2 = 2^2 E_1 = 391\text{ eV}$ and $E_3 = 3^2 E_1 = 881\text{ eV}$.

6. With $L = 1.2 \times 10^{-14}$ m = 10 fm,

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\text{ MeV}\cdot\text{fm})^2}{8(940\text{ MeV})(12\text{ fm})^2} = 1.4\text{ MeV}$$

7. (a) At $x = a$, $\psi_1 = \psi_2$ and $d\psi_1/dx = d\psi_2/dx$:

$$0 = (a-d)^2 - c \qquad \text{and} \qquad -2ab = 2(a-d)$$

From the second equation, $d = a(b+1)$. Inserting this into the first equation, we find $c = a^2b^2$.

(b) With $\psi_2 = \psi_3$ at $x = w$, we get $(w-d)^2 - c = 0$, or

$$w = d + \sqrt{c} = a(b+1) + \sqrt{a^2b^2} = a(2b+1)$$

The slope is discontinuous at w suggesting an infinite discontinuity in the potential energy at that location.

8. (a) The regions with $x < a$ and $x > a$ do not contribute to the normalization. The normalization integral is

$$\int |\psi(x)|^2\, dx = \int_{-a}^{+a} b^2(a^2 - x^2)^2\, dx = b^2 \int_{-a}^{+a} (a^4 - 2a^2x^2 + x^4)\, dx = b^2 \left(a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right)\Bigg|_{-a/2}^{+a/2}$$

Evaluating the integral and setting it equal to 1, we find

$$b^2 \left(2a^5 - \frac{4a^5}{3} + \frac{2a^5}{5} \right) = 1 \qquad \text{or} \qquad b = \sqrt{\frac{15}{16a^5}}$$

(b) $P(x)\, dx = |\psi(x)|^2\, dx = b^2(a^2 - x^2)^2\, dx$, and with $x = +a/2$ and $dx = 0.010a$ we obtain

$$P(x)\, dx = \frac{15}{16a^5} \left(\frac{a^2}{4} - a^2 \right)^2 (0.010a) = 0.0053$$

(c) $$P(a/2 : a) = \int_{a/2}^{a} |\psi(x)|^2\, dx = \int_{a/2}^{a} b^2(a^2 - x^2)^2\, dx = \frac{15}{16a^5} \left(a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right)\Bigg|_{a/2}^{a}$$

$$= \frac{15}{16a^5} \left[a^4 \left(a - \frac{a}{2} \right) - \frac{2a^2}{3} \left(a^3 - \frac{a^3}{8} \right) + \frac{1}{5} \left(a^5 - \frac{a^5}{32} \right) \right] = 0.104$$

9. With $\psi(x) = Cxe^{-bx}$, we have $d\psi/dx = Ce^{-bx} - bCxe^{-bx}$ and

$$\frac{d^2\psi}{dx^2} = -2bCe^{-bx} + b^2Cxe^{-bx}$$

We now substitute $\psi(x)$ and $d^2\psi/dx^2$ into the Schrödinger equation:

$$-\frac{\hbar^2}{2m}(-2bCe^{-bx}+b^2Cxe^{-bx})+U(x)Cxe^{-bx}=ECxe^{-bx}$$

Canceling the common factor of Ce^{-bx} and solving for E,

$$E=\frac{\hbar^2 b}{mx}-\frac{\hbar^2 b^2}{2m}+U(x)$$

The energy E will be a constant only if the two terms that depend on x cancel each other:

$$\frac{\hbar^2 b}{mx}+U(x)=0 \qquad \text{or} \qquad U(x)=-\frac{\hbar^2 b}{mx}$$

The cancellation of the two terms depending on x leaves only the remaining term for the energy:

$$E=-\frac{\hbar^2 b^2}{2m}$$

10. (a) The regions with $x < -L/2$ and $x > +L/2$ do not contribute to the normalization integral. The remaining integral is:

$$\int |\psi(x)|^2\,dx=\int_{-L/2}^{0}C^2(2x/L+1)^2\,dx+\int_0^{L/2}C^2(-2x/L+1)^2\,dx$$

$$=C^2\,(4x^3/3L^2+2x^2/L+x)\Big|_{-L/2}^{0}+C^2\,(4x^3/3L^2-2x^2/L+x)\Big|_0^{L/2}=C^2L/3$$

Setting the integral equal to 1 gives $C=\sqrt{3/L}$.

(b) $P(x)\,dx=|\psi(x)|^2\,dx=C^2(4x^2/L^2-4x/L+1)\,dx$ and with $x = 0.250L$ and $dx = 0.010L$,

$$P(x)\,dx=\frac{3}{L}\left(\frac{4}{L^2}\frac{L^2}{16}-\frac{4}{L}\frac{L}{4}+1\right)(0.010L)=0.0075$$

(c) $P(0:L/4)=\int_0^{L/4}|\psi(x)|^2\,dx=C^2\int_0^{L/4}\left(\frac{4}{L^2}x^2-\frac{4}{L}x+1\right)dx=\frac{3}{L}\left(\frac{4}{L^2}\frac{x^3}{3}-\frac{4}{L}\frac{x^2}{2}+x\right)\Bigg|_0^{L/2}=\frac{1}{2}$

(d) $\langle x\rangle=\int|\psi(x)|^2\,x\,dx=\frac{3}{L}\int_{-L/2}^{0}\left(\frac{4x^2}{L^2}+\frac{4x}{L}+1\right)x\,dx+\int_0^{L/2}\left(\frac{4x^2}{L^2}-\frac{4x}{L}+1\right)x\,dx$

$$=\frac{3}{L}\left(\frac{x^4}{L^2}+\frac{4x^3}{3L}+\frac{x^2}{2}\right)\Bigg|_{-L/2}^{0}+\frac{3}{L}\left(\frac{x^4}{L^2}-\frac{4x^3}{3L}+\frac{x^2}{2}\right)\Bigg|_{0}^{L/2}=0$$

It is apparent from the shape of the wave function that the equal probability densities for positive and negative x cancel to give an average of zero.

$$\langle x^2\rangle=\int|\psi(x)|^2\,x^2\,dx=\frac{3}{L}\int_{-L/2}^{0}\left(\frac{4x^2}{L^2}+\frac{4x}{L}+1\right)x^2\,dx+\int_{0}^{L/2}\left(\frac{4x^2}{L^2}-\frac{4x}{L}+1\right)x^2\,dx$$

$$=\frac{3}{L}\left(\frac{4x^5}{5L^2}+\frac{x^4}{L}+\frac{x^3}{3}\right)\Bigg|_{-L/2}^{0}+\frac{3}{L}\left(\frac{4x^5}{5L^2}-\frac{x^4}{L}+\frac{x^3}{3}\right)\Bigg|_{0}^{L/2}=\frac{L^2}{40}$$

The rms value is then $x_{\text{rms}}=\sqrt{\langle x^2\rangle}=\sqrt{L^2/40}=0.158L$.

11. (a) The normalization integral is

$$\int_{-\infty}^{+\infty}|\psi(x)|^2dx=\int_{-\infty}^{+\infty}A^2x^2e^{-2bx}dx=2A^2\int_{0}^{+\infty}x^2e^{-2bx}dx=2A^2\frac{2}{(2b)^3}=1$$

so $A=\sqrt{2b^3}$.

(b) The wave function can be written as $\psi(x)=Ae^{bx}$ for $x<0$ and $\psi(x)=Ae^{-bx}$ for $x>0$.

$$\int_{-\infty}^{+\infty}|\psi(x)|^2dx=\int_{-\infty}^{0}A^2e^{2bx}dx+\int_{0}^{+\infty}A^2e^{-2bx}dx=\frac{2A^2}{2b}$$

and $A=\sqrt{b}$.

12. For the $x>0$ wave function, we have

$$\frac{d\psi}{dx}=-Abe^{-bx}\qquad\text{and}\qquad\frac{d^2\psi}{dx^2}=Ab^2e^{-bx}$$

Substituting into the Schrödinger equation then gives

$$-\frac{\hbar^2}{2m}Ab^2e^{-bx}+UAe^{-bx}=EAe^{-bx}\qquad\text{or}\qquad-\frac{\hbar^2b^2}{2m}+U=E$$

This is consistent with a constant value of U, which we can take to be 0, giving $E=-\hbar^2b^2/2m$. Repeating the calculation for the $x<0$ wave function gives an identical result.

So the potential energy is a constant (zero) for $x < 0$ and for $x > 0$. What happens at $x = 0$? Note that the wave function is continuous at $x = 0$ (both give $\psi = A$ at $x = 0$) but that the derivative $d\psi/dx$ is not continuous. This suggests an infinite discontinuity in $U(x)$ at $x = 0$, and because the wave functions approach 0 as $x \to \infty$ there must be a negative potential energy that produces the bound states. So the potential energy is

$$U(x) = 0 \qquad x < 0,\ x > 0$$
$$U(x) = -\infty \qquad x = 0$$

This type of function is known as a delta function.

13. (a) $$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\ \text{eV}\cdot\text{nm})^2}{8(511{,}000\ \text{eV})(0.285\ \text{nm})^2} = 4.63\ \text{eV}$$

$4 \to 1$: $$\Delta E = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(4.63\ \text{eV}) = 69.5\ \text{eV}$$

(b) $4 \to 3$: $$\Delta E = E_4 - E_3 = 16E_1 - 9E_1 = 7E_1 = 7(4.63\ \text{eV}) = 32.4\ \text{eV}$$

$4 \to 2$: $$\Delta E = E_4 - E_2 = 16E_1 - 4E_1 = 12E_1 = 12(4.63\ \text{eV}) = 55.6\ \text{eV}$$

$3 \to 2$: $$\Delta E = E_3 - E_2 = 9E_1 - 4E_1 = 5E_1 = 5(4.63\ \text{eV}) = 23.2\ \text{eV}$$

$3 \to 1$: $$\Delta E = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(4.63\ \text{eV}) = 37.0\ \text{eV}$$

$2 \to 1$: $$\Delta E = E_2 - E_1 = 4E_1 - E_1 = 3E_1 = 3(4.63\ \text{eV}) = 13.9\ \text{eV}$$

14. With $E_1 = 1.54\ \text{eV}$ and $E_n = n^2 E_1$ we have

$$\Delta E_3 = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(1.54\ \text{eV}) = 12.3\ \text{eV}$$

$$\Delta E_4 = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(1.54\ \text{eV}) = 23.1\ \text{eV}$$

16. (a) $P(0:L/3)=\int_0^{L/3}|\psi_1(x)|^2\,dx=\int_0^{L/3}\frac{2}{L}\sin^2\frac{\pi x}{L}\,dx=\frac{2}{\pi}\int_0^{\pi/3}\sin^2 u\,du$

$$=\frac{2}{\pi}\left(\frac{u}{4}-\frac{\sin 2u}{4}\right)\Bigg|_0^{\pi/3}=0.1955$$

(b) $P(L/3:2L/3)=\int_{L/3}^{2L/3}|\psi_1(x)|^2\,dx=\int_{L/3}^{2L/3}\frac{2}{L}\sin^2\frac{\pi x}{L}\,dx=\frac{2}{\pi}\int_{\pi/3}^{2\pi/3}\sin^2 u\,du$

$$=\frac{2}{\pi}\left(\frac{u}{4}-\frac{\sin 2u}{4}\right)\Bigg|_{\pi/3}^{2\pi/3}=0.6090$$

(c) $P(2L/3:L)=\int_{2L/3}^{L}|\psi_1(x)|^2\,dx=\int_{2L/3}^{L}\frac{2}{L}\sin^2\frac{\pi x}{L}\,dx=\frac{2}{\pi}\int_{2\pi/3}^{\pi}\sin^2 u\,du$

$$=\frac{2}{\pi}\left(\frac{u}{4}-\frac{\sin 2u}{4}\right)\Bigg|_{2\pi/3}^{\pi}=0.1955$$

17. (a) $P(x)dx=|\psi_3(x)|^2\,dx=\frac{2}{L}\sin^2\frac{3\pi x}{L}\,dx=\frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.188\text{ nm})}{0.189\text{ nm}}0.001\text{ nm}=2.63\times10^{-5}$

(b) $P(x)dx=\frac{2}{L}\sin^2\frac{3\pi x}{L}\,dx=\frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.031\text{ nm})}{0.189\text{ nm}}0.001\text{ nm}=0.0106$

(c) $P(x)dx=\frac{2}{L}\sin^2\frac{3\pi x}{L}\,dx=\frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.079\text{ nm})}{0.189\text{ nm}}0.001\text{ nm}=5.42\times10^{-3}$

(d) A classical particle has a uniform probability to be found anywhere within the region, so $P(x)\,dx=(0.001\text{ nm})/(0.189\text{ nm})=5.29\times10^{-3}$.

18. With $E=E_0(n_x^2+n_y^2)$ the levels above $50E_0$ are as follows:

n_x	n_y	E	n_x	n_y	E
6	4	$52E_0$	6	5	$61E_0$
4	6	$52E_0$	5	6	$61E_0$
7	2	$53E_0$	7	4	$65E_0$
2	7	$53E_0$	4	7	$65E_0$
7	3	$58E_0$	8	1	$65E_0$
3	7	$58E_0$	1	8	$65E_0$

The level at $E = 65E_0$ is 4-fold degenerate.

19. With $E = E_0(n_x^2 + n_y^2/4)$ the levels are as follows:

n_x	n_y	E	n_x	n_y	E
1	1	$1.25E_0$	2	3	$6.25E_0$
1	2	$2.00E_0$	1	5	$7.25E_0$
2	1	$2.25E_0$	2	4	$8.00E_0$
1	3	$3.25E_0$	3	1	$9.25E_0$
2	2	$5.00E_0$	1	6	$10.00E_0$
1	4	$5.00E_0$	3	2	$10.00E_0$

The levels at $E = 5.00E_0$ and $E = 10.00E_0$ are both 2-fold degenerate.

20. Using Equations 5.39 and 5.40, we have

$$\frac{\partial \psi}{\partial x} = g(y)\frac{df}{dx} = g(y)(k_x A\cos k_x x - k_x B\sin k_x x)$$

$$\frac{\partial^2 \psi}{\partial x^2} = g(y)\frac{d^2 f}{dx^2} = g(y)(-k_x^2 A\sin k_x x - k_x^2 B\cos k_x x) = -k_x^2 g(y) f(x)$$

$$\frac{\partial \psi}{\partial y} = f(x)\frac{dg}{dy} = f(x)(k_y C\cos k_y y - k_y D\sin k_y y)$$

$$\frac{\partial^2 \psi}{\partial y^2} = f(x)\frac{d^2 g}{dy^2} = f(x)(-k_y^2 C\sin k_y y - k_y^2 D\cos k_y y) = -k_y^2 f(x) g(y)$$

With $U(x,y) = 0$ inside the well, Equation 5.37 gives

$$-\frac{\hbar^2}{2m}\left[-k_x^2 f(x)g(y) - k_y^2 f(x)g(y)\right] = Ef(x)g(y)$$

and so $E = \dfrac{\hbar^2}{2m}(k_x^2 + k_y^2)$.

21. Let $E_0 = \hbar^2\pi^2/2mL^2$. The energy states are then

n_x	n_y	n_z	E	degeneracy	n_x	n_y	n_z	E	degeneracy	n_x	n_y	n_z	E	degeneracy
1	1	1	$3E_0$	1	2	2	2	$12E_0$	1	1	1	4	$18E_0$	3
										1	4	1	$18E_0$	
1	1	2	$6E_0$	3	1	2	3	$14E_0$	6	4	1	1	$18E_0$	
1	2	1	$6E_0$		1	3	2	$14E_0$						
2	1	1	$6E_0$		2	1	3	$14E_0$		1	3	3	$19E_0$	3
					2	3	1	$14E_0$		3	1	3	$19E_0$	
1	2	2	$9E_0$	3	3	1	2	$14E_0$		3	3	1	$19E_0$	
2	1	2	$9E_0$		3	2	1	$14E_0$						
2	2	1	$9E_0$							1	2	4	$21E_0$	6
					2	2	3	$17E_0$	3	1	4	2	$21E_0$	
1	1	3	$11E_0$	3	2	3	2	$17E_0$		2	1	4	$21E_0$	
1	3	1	$11E_0$		3	2	2	$17E_0$		2	4	1	$21E_0$	
3	1	1	$11E_0$							4	1	2	$21E_0$	
										4	2	1	$21E_0$