Physics 218c

Dynamical Models - A Selective Study Part 1

(i)

⇒ Some Philosophy / OV

- there are a zillion models ..

- best to know the key models well

i.e. Most Models = key + $O(\epsilon)$

- two approaches:

a) Top → Down

Vlasov - Boltzmann

↓

Gyro/Drift Kinetics

↓

Fluids / GyroFluids

↓

Reduced Fluid
(i.e. Hasegawa - Wakatani)

~ IMHO, laborious

~ "systematic"

~ insight ?!

b) <u>Bottom-Up</u> ..

$\Uparrow$

Less reduced Models (4 field, 6 field) — Physical Ideas!

$\uparrow$

Reduced Models (HM) — but did one "leave out a term"?

$\uparrow$

Reduction Principle / Idea $\Leftarrow$ insight [time/space scale ordering]

Comment:

- historically, both top → down (Frieman, Rutherford; Frieman, Chen; Rosenbluth, Kadomtsev – Pogutse ..) and bottom-up (Hasegawa, et al, Kadomtsev, Jagdeev ...) evolved simultaneously
- synergism of both has been useful.
- here, will pursue:

  bottom → up: Drift Wave Models
  Reduced / Extended MHD

  Top → down: Gyrokinetics

- need 'speak several languages' → express same physics different way.

— Key Physics any way:

→ strong $\underline{B}_0$

→ $\omega < \Omega_i$, $\left.\begin{array}{l} l_\parallel \gg l_\perp \\ k_\parallel \ll k_\perp \end{array}\right\}$ anisotropy

→ $\lambda_D < l$,

$k_\perp \rho \lesssim 1$

in es: $\underline{v}_\perp \cong \frac{c}{B} \underline{E} \times \hat{z}$ $\qquad \hat{z} \cong \underline{\hat{c}_\parallel}$

→ consider electrostatics, then electromagnetics

(iii) On PV

→ What is PV? → $\left\{\begin{array}{l} \text{Conserved "effective charge" density} \\ \text{Generalized vorticity} \end{array}\right.$

n.b. "charge" = total charge (G.C. + polarization)

→ Why PV? → Conservation

→ mean-fluctuation exchange.

→ PV is macroscopic (unlike f) yet conserved

and → Point of H-M is that Drift Wave Turbulence is 'like' Geophysical Fluid Turbulence

and → GFD makes heavy use of PV.

indeed: "GFD = Fluid Dynamics of PV" Likewise plasma ...

Key Point: $Ro \ll 1$

↓

Rossby #

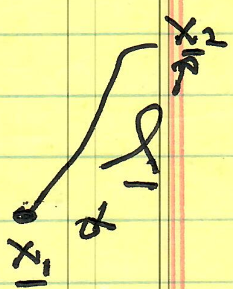
$Ro \sim V_\perp / L_\perp \Omega$ (Fluid)

$Ro \sim V_\perp / L_\perp \Omega_i$ (Plasma)

@ 2D flow

⇒ vorticity along rotation axis is key

→ what?



$$\frac{d\underline{x}}{dt} = \underline{V}(\underline{x})$$

"points frozen into flow"

Can consider $\underline{\ell}$, a line segment $\underline{\ell}$, (i.e. flexible monofilament inserted into the flow ...)

$$\frac{d}{dt}\underline{\ell} = \underline{V}(\underline{x}_2) - V(\underline{x}_1)$$

$(\underline{x}_2 - \underline{x}_1)/2 \equiv \underline{\ell}$

so

$$\boxed{\frac{d\underline{\ell}}{dt} = \underline{\ell} \cdot \nabla \underline{V}}$$

$\underline{\ell}$ "frozen-in" to the flow.

N.B. :

→ $\frac{d\underline{\ell}}{dt} \neq 0$ ⇒ allows for stretching

→ obvious similarity to $\underline{B}/\rho$ in ideal MHD (local form, Alfven's Theorem),

$$\boxed{\frac{d}{dt} \underline{B}/\rho = \frac{\underline{B}}{\rho} \cdot \nabla \underline{V}}$$

$\frac{\underline{B}}{\rho}$ "frozen in"

eqns. $\underline{\ell}$, $\underline{B}/\rho$ same form

→ For flow (i.e. ions, for plasma)

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\frac{\nabla P}{\rho} - \underbrace{2\underline{\Omega} \times \underline{V}}_{\text{Coriolis / Lorentz}}$$

Then seek vorticity

(Plasma / GFD all about vorticity)

Why? → vorticity along $\langle \underline{B} \rangle$ describes dynamics

→ ~ 2D.

$$\underline{\omega} = \underline{\nabla} \times \underline{V}$$

then →
$$\underline{V} \cdot \underline{\nabla}\, \underline{V} = \underbrace{-\underline{\nabla}\left(\frac{V^2}{2}\right)}_{\text{dynamic pressure}} \underbrace{- \underline{V} \times \underline{\omega}}_{\text{Magnus Force}}$$

→ $P = P(\rho)$

otherwise, enter Ertel's Thm. and non-conservation of circulation

i.e. $P(\rho, T) \rightarrow \nabla\rho \times \nabla T$ drive

see Muller. (good Paper Topic)

then,

$$\partial_t (\underline{\omega} + 2\underline{\Omega}) = \underline{\nabla} \times \left[\underline{V} \times (\underline{\omega} + 2\underline{\Omega})\right]$$

and, since $\underline{\Omega}$ $\begin{cases} \text{static} \\ \text{uniform} \end{cases}$ $\underline{\Omega} = \Omega \hat{z}$

$$+ \quad \frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{V}) = 0$$

⇒ Freezing in law for vorticity:

$$\frac{d}{dt}\frac{(\underline{\omega}+2\underline{\Omega})}{\rho} = \frac{(\underline{\omega}+2\underline{\Omega})}{\rho}\cdot\underline{\nabla}\,\underline{V}$$

obviously: $\Updownarrow$

$$\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \left(\frac{\underline{B}}{\rho}\right)\cdot\underline{\nabla}\,\underline{V}$$, Ideal MHD $\underline{E}+\frac{\underline{V}\times\underline{B}}{c}=0$

so $\frac{\underline{\omega}+2\underline{\Omega}}{\rho}$ frozen-in!

N.B. For $\begin{cases}|\underline{\Omega}| \gg \text{other rates in problem} \\ \rho \simeq \text{const}\end{cases}$

⇒ $\underline{\Omega}\cdot\underline{\nabla}\,\underline{V} \cong 0$ ⇒ Taylor-Proudman Theorem

i.e. flow uniform along direction of rotation axis

⇒ 2D-ized!

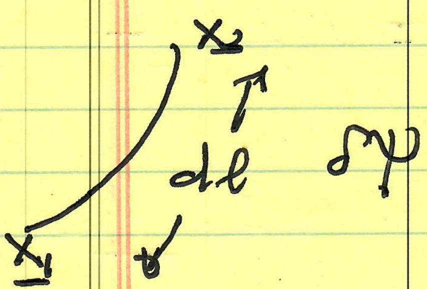
N.B.: Obviously frozen-in ≠ passive!

Now, to PV!

- consider a passive scalar ψ (inviscid)!

$$\frac{d}{dt}\psi = 0$$

(ψ conserved along trajectory)



$$\delta\psi = \psi(2) - \psi(1) \cong \underline{\nabla}\psi \cdot d\underline{l}$$

$$\frac{d}{dt}\delta\psi = 0$$

so

$$\boxed{\frac{d}{dt}\left(\underline{\nabla}\psi \cdot d\underline{l}\right) = 0}$$

re-label, only:
l infinitesimal $\Rightarrow$
increment
$\underline{l} \to d\underline{l}$

but

$$\frac{d\,d\underline{l}}{dt} = d\underline{l}\cdot\underline{\nabla}\,\underline{V}$$

$$\Updownarrow$$

$$\frac{d}{dt}\left(\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}\right) = \left(\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}\right)\cdot\underline{\nabla}\,\underline{V}$$

so $d\underline{\ell} \leftrightarrow \dfrac{\underline{\omega} + 2\underline{\Omega}}{\rho}$

$\Rightarrow$
$$\boxed{\frac{d}{dt}\left[\frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \underline{\nabla}\psi}{\rho}\right] = 0}$$

$\rightarrow$ statement of PV conservation

$$\boxed{PV = \rho_q = \frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \underline{\nabla}\psi}{\rho}}$$

"charge" effective density

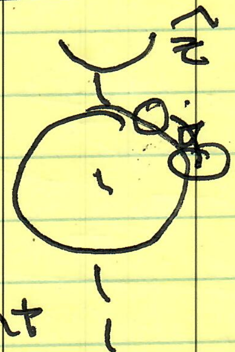
ρ_q notation is confusing, so

$$\boxed{q = \frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \underline{\nabla}\psi}{\rho}}$$ $\underset{\sim}{PV}$

$\underset{\sim}{n.b.}$: ψ $\underset{\sim}{\text{any}}$ conserved scalar !

Comments on PV:

$$q = \frac{(\underline{\omega} + 2\underline{\Omega})\cdot\underline{\nabla}\psi}{\rho}$$



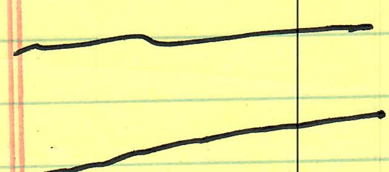
$\underline{\nabla}\psi \equiv \hat{z}$

i) utility

→ displace fluid element in latitude

⇒ $\underline{\omega}\cdot\hat{z}$ must change

⇒ flow change predicted w/o detailed calculation

→ 

shallow water

$\rho \to H$.

displace in density, thickness,

⇒ flow changes.

ii.) Conservation $\Leftrightarrow$ Symmetry (Noether)

Particle re-labeling: $\underline{x}\,(\underline{s}, \tau)$

(Lagrangian)

if $s \to s' = s + \delta s$ (re-labeling)

and thermodynamic state invariant

$\Rightarrow$ PV conserved. (see Muller).

iii.) Related: Kelvin's Thm.

have vorticity / ρ freezing in:

$$\frac{d}{dt}\frac{\underline{\omega}}{\rho} = \frac{\underline{\omega}}{\rho}\cdot\underline{\nabla}\,\underline{V},$$

and with $\underline{\omega}$

$$\frac{d}{dt}\left(\frac{\underline{\omega}+2\underline{\Omega}}{\rho}\right) = \frac{(\underline{\omega}+2\underline{\Omega})}{\rho}\cdot\underline{\nabla}\,\underline{V}$$

vorticity induction

$$\Leftrightarrow \quad \frac{\partial}{\partial t}(\underline{\omega} + 2\underline{\Omega}) = \underline{\nabla}\times\underline{V}\times(\underline{\omega}+2\underline{\Omega})$$

and recall for $\underline{B}$

$$\partial_t \underline{B} = \underline{\nabla}\times(\underline{V}\times\underline{B})$$

$\rightarrow$ induction eqn.

and recall from ideal MHD:

$$\frac{d}{dt}\left(\int d\underline{a}\cdot\underline{B}\right)=0$$

flux conserved.

So, obvious here that:

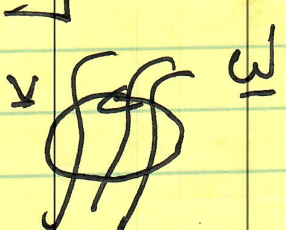
$$\frac{d}{dt}\left[\int d\underline{a}\cdot(\underline{\omega}+2\underline{\Omega})\right]=0.$$

Kelvin's Thm.

n.b. $\underline{\Omega}=0$

$$\frac{d}{dt}\left[\int d\underline{a}\cdot\underline{\omega}\right]=\frac{d}{dt}\oint d\underline{l}\cdot\underline{v}=0$$

↳ circulation



usual form.

Kelvin's Thm: Total circulation (parcel + planetary) conserved.

→ Kelvin's Thm → Charney Eqn →

CHM (Hasegawa Mima) Eqn

N.B. There is an important disconnect between Charney and Hasegawa-Mima

Then

$$\boxed{\oint d\underline{\ell}\cdot(\underline{\omega} + 2\underline{\Omega}) = \text{Const}}$$

"Its all Kelvin's Thm"

$$\frac{d}{dt}\,\text{Const} = 0$$

$Ro \ll 1$

$$\rho \frac{d\underline{v}}{dt} = -\nabla p - 2\underline{\Omega}\times\underline{v}$$

$\hat{c} = \hat{z}$

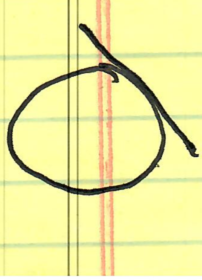
⊙ 2D dynamics

so
$$\underline{v} \cong -\frac{\nabla_\perp p \times \hat{c}}{2\Omega}$$

⇒ "Geostrophic balance"

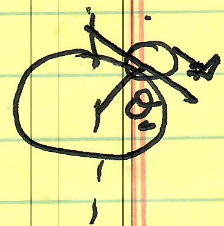
stream function $\phi \Leftrightarrow p$.

why flow circulates around highs, lows



tangent plane ⇒ "β-plane"

$\beta \ni P_{on} / P_B$!



$$\frac{d\omega}{dt} = -\frac{2\Omega}{\bar{A}} \sin\theta_0 \frac{dA}{dt}$$

changing projected area.

$$= -\frac{2\Omega}{\bar{A}} \sin\theta_0 \frac{d\theta}{dt}$$

$\beta \equiv$ gradient of coriolis parameter

$$= -\beta v_y$$

$\beta \equiv 2\Omega \sin\theta_0 / R$

~ grad. (Coriolis force)

$\omega = \nabla^2 \phi$

so, at long last:

$$\boxed{\begin{gathered} \frac{d}{dt}(\omega + \beta y) = 0 \\ \frac{d}{dt} = \partial_t + \underline{v} \cdot \underline{\nabla} \end{gathered}}$$

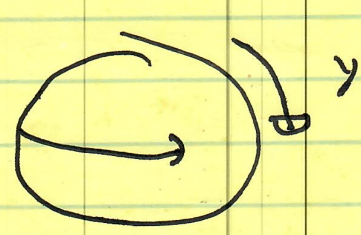
[inviscid Charney Equation

$\underline{v} = -\underline{\nabla} p \times \hat{z} / 2\Omega$

$$PV = \underbrace{\omega}_{\text{parcel}} + \underbrace{\beta y}_{\text{planetary}}$$

Latitudinal displacement → change in relative vorticity

Linear Consequence:



$$\partial_t \nabla^2 \phi = -\beta \partial_x \phi$$

$$\boxed{\omega = -\beta k_x / k^2}$$ $\Rightarrow$ Rossby wave (azimuthally asymmetric vortex mode)

and

$$\boxed{\begin{array}{l} k_x = 0 \;\Rightarrow\; \text{azimuthal symmetry} \\ \omega = 0 \;\Rightarrow\; \text{zonal flow} \end{array}}$$

(vortex mode)

N.B.:

$\rightarrow$ $v_{gr\,y} = 2\beta \dfrac{k_x k_y}{(k^2)^2}$ $\rightarrow$ $\langle v_y v_x \rangle$

$\rightarrow$ Rossby wave propagation intimately connected to Reynolds stress and momentum flux!

$\rightarrow$ Latitudinal PV energy flux

$\Rightarrow$ change in circulation

→ Now, isn't this class about Plasmas ?

well ....

$$q = \frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \underline{\nabla} \psi}{\rho} \Rightarrow PV.$$

now $\underline{\Omega} \leftrightarrow \underline{\Omega}_i$

$$\left.\begin{aligned} 2\underline{\Omega} &\to \Omega_i \hat{z} \\ \rho &\to n_0(r) + \tilde{n} \\ \underline{\nabla}\psi &\to \hat{z} \end{aligned}\right\}$$

$\Rightarrow$

$$\frac{d}{dt}\left[\frac{\omega_z + \Omega_i}{n_0(r) + \tilde{n}}\right] = 0$$

so

$$\boxed{\frac{d}{dt}\hat{\omega}_z - \frac{\Omega_i}{n_0}\frac{d\tilde{n}}{dt} = 0}$$

now :
$$\omega_z = (c/B_0)\nabla_\perp^2 \phi$$

$$\underline{V} = -\frac{c}{B}\underline{\nabla}\phi \times \hat{z}$$

$\omega / k_{\parallel} v_{the} \to 0$

Now, also: $v_{thi} < \dfrac{\omega}{k_{\parallel}} < v_{the}$

(c/c' con-acoustic)

$$\frac{\tilde{n}_i}{n_0} = \frac{\tilde{n}_e}{n_0} \cong \frac{|e|\phi}{T_e}$$

$\Rightarrow$ 3D Physics in 2D Eqn !

$$\boxed{\frac{d}{dt}\left(\frac{|e|\phi}{T} - \rho_s^2 \nabla_\perp^2 \frac{|e|\phi}{T}\right) + v_* \partial_y \frac{|e|\phi}{T} = 0}$$

$\boxed{\underline{\text{Hasegawa}} - \underline{\text{Mima}}\ \underline{\text{Eqn}}.\ !}$

$\rho_s^2 = c_s^2 / \Omega_c^2$

$v_* = \dfrac{\rho_s}{L_n} c_s$ $\qquad 1/L_n = -\dfrac{1}{n_0}\dfrac{\partial n_0}{\partial r}$

Diamagnetic Velocity

$\to$ H-M eqn $\equiv$ PV conservation $\boxed{\text{unusual derivation}}$

$\updownarrow$

i.e.

$$\frac{d}{dt}\left(\phi - \rho_s^2 \nabla_\perp^2 \phi + \ln n_0(r)\right) = 0$$

$\to$ Entry point to Zoology of Drift-Wave Systems

→ Usual Derivation: (see H+M, 1978)

$$\frac{\partial n}{\partial t} + \underline{\nabla}\cdot(n\underline{V}) = 0$$

$$\partial_t \tilde{n} + \tilde{V}_r \partial_r n_0 + \underline{\tilde{V}}\cdot\nabla\tilde{n} + n_0 \underline{\nabla}\cdot\underline{\tilde{V}} = 0$$

$$\frac{\tilde{n}}{n_0} = \frac{|e|\tilde{\phi}}{T_e}$$

$$\underline{V} = -\frac{c}{B}\underline{\nabla}\phi\times\hat{z} + \rho_s^2 n_0 \frac{d}{dt}\underline{E}_\perp$$

↓ l.o. $\underline{E}\times\underline{B}$ (indep of charge, mass!)

↓ 1st order ω/Ω_i Polarization → mass dependent

c.f. Landau, Lifshitz "Classical Theory of Fields" or basic plasma book

$$\underline{\nabla}\cdot\underline{\tilde{V}} = \underline{\nabla}\cdot V_{pol} + \underline{\nabla}\cdot \cancel{V_{E\times B}}$$

$$= +\rho_s^2 n_0\, \underline{\nabla}\cdot\frac{d}{dt}\underline{E}_\perp$$

⇓

NL polarization.

$\tilde{V} \cdot \nabla \tilde{n} = 0$

$\Rightarrow$

$$\left(\partial_t - \frac{c}{B_0} \nabla\phi \times \hat{z} \cdot \nabla\right)\left(\phi - \rho_s^2 \nabla_\perp^2 \phi\right) + V_* \frac{\partial \phi}{\partial y} = 0$$

H-M.

$\Rightarrow$ 2D fluid: $\partial_t \underline{\omega} = \nabla \times \underline{V} \times \underline{\omega}$

$$\partial_t \omega + \underline{V} \cdot \nabla \omega = \cancel{\underline{\omega} \cdot \nabla \underline{V}}$$

$\Rightarrow$ Linear wave $\Rightarrow$ Electron Drift Wave!

$$\omega = k_\theta V_* / \underbrace{1 + k_\perp^2 \rho_s^2}$$

$\Rightarrow$ no growth

(a) wave.

generally $\omega < \omega_*$

$\Rightarrow$ (1) important for drift instability

(a) inertia of ions

N.B. Rossby Wave:

$$\omega = -\beta k_x / k^2$$

(2) Radial propagation

$\Rightarrow$ Flows

$\Leftrightarrow$ ion inertia.

"1" is from Boltzmann electrons.

No counterpart in fluids!

→ Now, what of Zonal Flow?

Naively:

$$\boxed{\omega = k_\theta v_* / 1 + k_\perp^2 \rho^2 = k_\theta v_* / 1 + k_\perp^2 \rho_s^2}$$

so $\omega \to 0$ for azimuthal symmetry

⇒ zonal flow...?

But recall used

$v_{Thi} < \frac{\omega}{k_\parallel} < v_{The}$ for H-M.

but Z.F. exhibits 2 symmetries

→ poloidal → $k_\theta = 0$

→ toroidal → $k_z = 0$

⇒ $k_\parallel = 0$

so electrons not Boltzmann

What to do?

Recall: QN $\Leftrightarrow$ $\underline{\nabla}\cdot\underline{J} = 0$

$$\underline{J} = \underline{J}_\perp + \underline{J}_\parallel$$

$$\underline{\nabla}\cdot\underline{J} = \underbrace{\cancel{\nabla_\parallel J_\parallel}}_{\substack{k_\parallel \to 0 \\ \tilde{B}_r \to 0}} + \underline{\nabla}_\perp\cdot\underline{J}_\perp = 0$$

$\Rightarrow$ So ZF equation: $\underline{\nabla}_\perp\cdot\underline{J}_\perp = 0$

$$\underline{J}_\perp = \underbrace{\cancel{\underline{J}_{E\times B}}}_{0} + \underline{J}_{pol} \qquad \text{(ignore curvature)}$$

$$\underline{J}_{E\times B} = 0 \quad \text{as} \quad \underline{V}_{E\times B\,i} = \underline{V}_{E\times B\,e}$$

$$n_e = n_i$$

$$\underline{J}_{pol} = \underbrace{\underline{J}_{pol\,i}}_{\rho_i^2 \sim m_i} + \underbrace{\underline{J}_{pol\,e}}_{\rho_e^2 \sim m_e} \qquad \rho_e = v_{Te}^2/\Omega_e^2$$

so $\underline{J}_{pol} = \underline{J}_{pol\,c}$

and so:

Zonal Flow: $\underline{\nabla} \cdot \underline{J}_{pol\,c} = 0$

$$\Rightarrow \boxed{\frac{d}{dt} \rho_s^2 \nabla_\perp^2 \phi = 0}$$

→ akin 2D Fluid

→ Important: ZF $\underline{\underline{\text{not}}}$ governed by same eqn. as drift wave

Contrast to Rossby / Charney Eqn

"CHM equation" is misnomer.

→ History:

- Charney ~ 1958 ⊛

20 yr lead

- H-M → 1978

- Sagdeev, Shapiro, Schevchenko → 1978 (equivalent)

also

- Hasegawa and Kodama → 1978.

N.B. All missed confinement implications till H-mode discovered in 1982.

→ Why important.

ZF ↔ symmetry ↓ $\langle\ \rangle = \langle\ \rangle_{y,z}$

$$\partial_t \langle \nabla^2 \phi \rangle = -\partial_r \langle \tilde{v}_r \nabla^2 \tilde{\phi} \rangle$$

avg of vorticity flux due waves drives Z.F.

"inertia" ~ $k_r^2 \rho_s^2$

Compare wave inertia:

$$\partial_t \left(\phi - \rho_s^2 \nabla^2 \tilde{\phi} \right) = - V_* \partial_y \phi$$

$$\sim \underline{1} + k_\perp^2 \rho_s^2$$

$\hookrightarrow$ dominant, as $k_\perp^2 \rho_s^2 \ll 1$

$\Rightarrow$

- Zonal modes more easily excited in plasma, than fluid

- "Zonal modes are modes of minimal inertia" ...

  PD, Itoh, Itoh, Hahm 2005