Physics 218c

Models 1b — PV and Drift Waves, 2

→ Hasegawa–Wakatani →

Drift-Alfven ← Reduced MHD

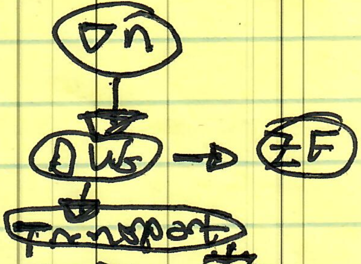
Why?

- Hasegawa-Wakatani is proto-type of drift instability / relaxation system

i.e. { vorticity / PV
+
density, temperature

also illustrates connection:

relaxation ↔ Zonal flow
branching ratio.



- If understand H-W well, then easy to grasp:
  - collisionless drift wave, a.k.a. "Universal Mode"
  - DTEM, CTEM etc.

- Reduced MHD is prototype of electromagnetic system.

  also illustrates elimination of fast mode ⇒ model reduction.

- Drift-Alfvén unifies HW + RMHD

N.B.

① → for complete zoology, see Kadomtsev + Pogutse 1970.
Good OV of models, modes. Weak on nonlinear evolution

② History:

→ Sagdeev and Moiseev '60's first developed theory of collisional drift wave (i.e. Hasegawa-Wakatani)

→ Reduced MHD often referred to as "Strauss Equations", after Strauss, '76 (posted). But:

- Rosenbluth, et. al. '74
- Kadomtsev and Pogutse '73 (?)

⇒ clearly many origins.

③ Re: ITG (Ion temperature Gradient) modes

⇒ Physics slightly different from (electron) drift waves

⇒ Coming attraction.

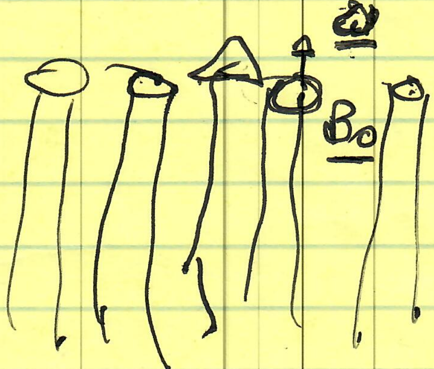
Proceeding ....

$$\left\{ Ro = V_\perp / \ell_\perp \Omega << 1 \right.$$

defines PV.

Recall:

$$PV = q = \frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \underline{\nabla} \psi}{\rho}$$



$$\frac{dq}{dt} = 0 \qquad (PV \text{ conserved})$$

⇒

$$\frac{d}{dt}\left[\frac{\omega_z + \Omega_i}{n_0(r) + \tilde{n}}\right] = 0$$

$$\underline{V} = -\frac{c}{B_0} \underline{\nabla}\phi \times \hat{z}$$

$$\omega_z = (c/B_0) \nabla_\perp^2 \phi$$

so

$$\frac{d}{dt}\hat{\omega}_z - \Omega_i \frac{1}{n_0}\frac{d\tilde{n}}{dt} = 0$$

Ertel's Thm

If $p \neq p(\rho)$ / non-isentropic :

⇒ → clearly complicates K-Thm.

$$\frac{\partial \underline{\omega}}{\partial t} = \underbrace{\underline{\nabla}\times(\underline{v}\times\underline{\omega})}_{\text{before}} + \underbrace{\frac{1}{\rho^2}\underline{\nabla}\rho\times\underline{\nabla}p}_{\text{from } -\underline{\nabla}\times\frac{\underline{\nabla}p}{\rho}}$$

$\underline{\omega} \to \underline{\omega} + 2\underline{\Omega}$

Now $\left(\begin{matrix}\text{specific}\\ \text{entropy}\end{matrix}\right)$ $s = s(p,\rho)$ → $\underline{\nabla}s\cdot\underline{\nabla}\rho\times\underline{\nabla}p = 0$

$\underline{\nabla}s \cdot$ ω equation ⇒

$$\begin{aligned}
\underline{\nabla}s\cdot\frac{\partial\underline{\omega}}{\partial t} &= \underline{\nabla}s\cdot\underline{\nabla}\times(\underline{v}\times\underline{\omega}) \\
&= -\underline{\nabla}\cdot\left[\underline{\nabla}s\times(\underline{v}\times\underline{\omega})\right] \\
&= -\underline{\nabla}\cdot\left[\underline{v}(\underline{\omega}\cdot\underline{\nabla}s)\right] + \underline{\nabla}\cdot\left[\underline{\omega}(\underline{v}\cdot\underline{\nabla}s)\right] \\
&= -(\underline{\omega}\cdot\underline{\nabla}s)\,\underline{\nabla}\cdot\underline{v} - \underline{v}\cdot\underline{\nabla}(\underline{\omega}\cdot\underline{\nabla}s) \\
&\quad + \underline{\omega}\cdot\underline{\nabla}(\underline{v}\cdot\underline{\nabla}s)
\end{aligned}$$

but

$$\frac{\partial S}{\partial t} + \underline{V}\cdot\underline{\nabla} S = 0$$

(ideal)
(broken by thermal diffusion)

i.e. entropy evolves, but conserved along fluid trajectories

so

$$\partial_t \left(\underline{\omega}\cdot\underline{\nabla} S\right) + \underline{V}\cdot\underline{\nabla}\left(\underline{\omega}\cdot\underline{\nabla} S\right) + \left(\underline{\omega}\cdot\underline{\nabla} S\right)\underline{\nabla}\cdot\underline{V} = 0$$

$\uparrow$
continuity

$$\boxed{\frac{d}{dt}\left(\frac{\underline{\omega}\cdot\underline{\nabla} S}{\rho}\right) = 0}$$

Ertel's Thm

obviously:

$$\frac{\underline{\omega}\cdot\underline{\nabla} S}{Z} = PV,$$

with

$S = \psi$

$$\frac{dS}{dt} = 0$$

N.B.: $\left\{\right.$ PV conserved for non-isentropic ideal fluid

but Kelvin's Thm does not apply.

(also MHD)

$\tilde{n}_i = \tilde{n}_e = n_0 \frac{|e|\hat{\phi}}{T}$ $v_{Ti} < \frac{\omega}{k_\parallel} < v_{Te}$

⇒ H-M:

$$\frac{d}{dt}\left(\frac{|e|\hat{\phi}}{T} - \rho_s^2 \nabla_\perp^2 \frac{|e|\hat{\phi}}{T}\right) + V_* \partial_y \frac{|e|\hat{\phi}}{T} = 0$$

$$\frac{d}{dt}\left[\left(\phi - \rho_s^2 \nabla^2 \phi\right) + \ln n_0\right] = 0$$

→ or, $\underline{\nabla} \cdot \underline{J} = 0$

→ stable drift waves $\omega = \omega_{*e} / 1 + k_\perp^2 \rho_s^2$

and

⇒ Zonal Flow   (c.f. Charney)

→ $\omega = 0$ mode,
poloidal) symmetry
toroidal

$$\underline{\nabla}_\perp \cdot \underline{J}_\perp = \underline{\nabla}_\perp \cdot \underline{J}_{pol} = 0$$

$$\frac{d}{dt} \rho_0 \nabla_r^2 \phi = 0 \quad ; \quad \underset{\text{only}}{\phi = \phi(r)}$$

n.b. to connect to PV; (c.f. Wehed Question)

recall: $\frac{d}{dt} \hat{\omega}_z - \Omega_0 \frac{1}{n_0} \frac{d\hat{n}}{dt} = 0$

so if allow <u>pure</u> vortex mode

<u>i.e.</u> $\tilde{n} \to 0$, $\omega_z \neq 0$

n.b. $k_\parallel \to 0$

$\cancel{\frac{\omega}{k_\parallel} < v_{the}}$

with poloidal, toroidal symmetry

<u>i.e.</u> $$\frac{d\tilde{n}}{dt} = \overset{ⓥ}{\cancel{\frac{\partial \tilde{n}}{\partial t}}} + \underset{\text{symmetry}}{\cancel{\tilde{v}_r \frac{d n_0}{dr}}} + \overset{ⓥ}{\cancel{\underline{\tilde{v}}_\perp \cdot \underline{\nabla} \tilde{n}}}$$

then vortex mode obeys;

$$\boxed{\frac{d}{dt}\omega_z = \frac{d}{dt}\nabla_\perp^2 \phi = 0}$$

as above.

N.B. <u>Can</u> reconcile PV approach with ZF.

Observe:

→ Zonal flow cannot tap/relax free energy source

$\partial_y = 0 \to \tilde{v}_r \to 0$

so can't relax $n_0(r)$, $T_0(r)$ etc.

→ then zonal flow excited by nonlinear interactions only. $\phi_{ZF} = \phi(r)$

i.e.

$$\boxed{\partial_t \nabla_r^2 \phi_Z = -\partial_r \langle \tilde{V}_r \nabla_\perp^2 \tilde{\phi} \rangle_Z + \ldots}$$

i.e. → vorticity flux drives zonal flow.

→ ZF is electrostatic potential fluctuation. ⇒ "E×B flow"

Distinct from physical mass flow.

Careful with "flow"!

→ Why care about ZF?

⇒ shearing

TBC ⇒ energy storage → won't cause transport.

→ ● Hasegawa-Wakatani (<u>Next</u> most simple!)

→ HM supports stable waves, only.

→ How find instability?

Clue: <u>Flux</u>.

In particular, ∇n drive ⇒

$$\langle \tilde{v}_r \tilde{n} \rangle \neq 0 \qquad \underline{\text{transport}}!$$

In drift <u>wave</u>,

$$\tilde{n} = n_0 |e| \tilde{\phi} / T \qquad \text{so}$$

$$\langle \tilde{v}_r \tilde{n} \rangle = \langle \partial_y \tilde{\phi} \tilde{\phi} \rangle \to 0.$$

⇒ look for way to <u>weaken</u> <u>coupling</u> of $\tilde{n}, \tilde{\phi}$!

⇔ Phase shift ⇒ fundamental to all electron drift waves.

How?

Evolve density with dissipative phase lag!

⇒ Model

$$\underline{\nabla}\cdot\underline{J} = 0$$

$$\underline{\nabla}_\perp\cdot\underline{J}_\perp + \nabla_\parallel J_\parallel = 0$$

$$\underline{J}_\perp = \cancelto{0}{\underline{J}_{\underline{E}\times\underline{B}}} + \underline{J}_{pol} + \underbrace{\underline{J}_{ps}}_{\substack{\text{curvature}\\ \text{(neglect)}}} \Rightarrow \text{interchange drive}$$

and

$$\nabla_\parallel = \nabla_\parallel^{(0)} + \cancelto{\text{e.s.}}{\frac{\tilde{\underline{B}}_\perp}{B_0}\cdot\underline{\nabla}_\perp}$$

$$\nabla_\parallel^{(0)} = \frac{\underline{B}_0\cdot\underline{\nabla}}{|B_0|}$$

$$\underline{\nabla}_\perp\cdot\underline{J}_{\perp\,pol} + \nabla_\parallel^{(0)}\tilde{J}_\parallel = 0.$$

but we know:

$$\underline{\nabla}_\perp\cdot\underline{J}_{\perp\,pol} = c^2 \underbrace{\frac{d}{dt}}_{\partial_t + \underline{V}_{E\times B}\cdot\underline{\nabla}} \nabla_\perp^2\phi$$

current, strictly

$$\underline{\nabla}_\perp \cdot \underline{J}_{\perp pol} = \rho_s^2 \frac{d}{dt} \nabla_\perp^2 \phi = -\nabla_\parallel^{b)} \hat{J}_\parallel$$

What is $\hat{J}_\parallel$? → (~) controlled by electrons

parallel current.

$$\hat{J}_\parallel = -n_0|e| \left(\underset{\checkmark}{\hat{V}_{\parallel e}} - \underbrace{\hat{V}_{\parallel i}} \right) \simeq -n_0|e| \hat{V}_{\parallel e}$$

ohm's Law (general) → acoustic wave coupling.

For $\hat{V}_{\parallel e}$, illuminating to examine Drift Kinetic Equation (simple) for electrons

$$\frac{\partial f}{\partial t} + V_\parallel \hat{n} \cdot \nabla f - \frac{c}{B} \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla} f - \frac{|e|}{m_e} E_\parallel \frac{\partial f}{\partial v_\parallel} = C(f)$$

$$\frac{d\underline{x}_\perp}{dt} = \underline{V}_{\perp E\times B}$$

$$\frac{dx_\parallel}{dt} = V_\parallel \qquad \frac{dV_\parallel}{dt} = -\frac{|e|}{m_e} \underset{(-\nabla_\parallel \phi)}{E_\parallel}$$

obviously $\underline{\nabla} \cdot \underline{V}_D = 0$.

N.B. Electron inertia small ⇒ stay on the field line .... easy

Then $V_\parallel = \int d^3\underline{v}\, v_\parallel f$

$$\frac{\partial \tilde{V}_\parallel}{\partial t} + n\cdot\nabla \int d^3v\, v_\parallel^2 f - \frac{c}{B}\nabla_\perp\phi\times\hat{z}\cdot\nabla\tilde{V}_\parallel$$

advection of current

$$+\frac{|e|}{m_e}E_\parallel = -\nu_{ei}\tilde{V}_\parallel$$

frictional losses ⇒ loss (momentum conservation!)

so

$$\frac{m_e}{|e|}\left[\frac{\partial}{\partial t}\hat{V}_\parallel - \frac{c}{B_0}\nabla_\perp\phi\times\hat{z}\cdot\nabla_\perp\hat{V}_\parallel\right]$$

↑ inertia → small

→ 0

$$+\frac{m_e}{|e|}\nabla_\parallel\underbrace{\langle v_\parallel^2 f\rangle}_{\nabla_\parallel p_e}$$

→ indep m_e.

$$+E_\parallel = -\frac{\nu_{ei}m_e}{|e|}\hat{V}_\parallel$$

Ohm's Law

$$\cong +\underbrace{\frac{\nu_{ei}m_e}{n_0|e|^2}}_{\eta}\tilde{J}_{\parallel\phi}$$

→ parallel electron momentum.

η → resistivity

Then have:

$$\frac{\nu_{ei} m_e}{n_0 |e|^2} \tilde{J}_{\parallel e} = -\nabla_\parallel \phi + \frac{m_e}{|e|} \underbrace{\langle v_\parallel^2 f \rangle}_{P_e}.$$

⇒ Electron pressure contribution to Ohm's Law is complex

→ Thermal force, Time-dependent thermal force .... $\Downarrow$ Microtearing (MTM)

⇒ <u>Simplest</u> → Isothermal

$$P = \underset{\text{const.}}{T}\, n$$

$$\boxed{\tilde{J}_{\parallel e} = -\underbrace{\frac{v_{The}^2}{\nu_{e,i}}}_{D_\parallel} \nabla_\parallel \left(\tilde{\phi} - T \frac{\tilde{n}}{n_0} \right)}$$

n.b. Obviously, $l_{mfp} < l_\parallel \sim R_\Sigma$,

but not all.

Then,

$$\boxed{\rho_s^2 \frac{d}{dt}\nabla_\perp^2\tilde{\phi} = D_\parallel \nabla_\parallel^2\left(\tilde{\phi} - T\frac{\tilde{n}}{n_0}\right)}$$

HW 1

Obviously, need n equation.
Use $\tilde{n}_e$ (as $\tilde{n}_i = \tilde{n}_e$), since
electrons on lines.

So $$\frac{\partial n}{\partial t} + \underline{\nabla}\cdot(n\underline{V}) = 0$$

electron flow, strictly.

$$\Rightarrow \frac{\partial \tilde{n}}{\partial t} + \tilde{V}_r\frac{\partial n_0}{\partial r} + n_0\nabla_\parallel \tilde{V}_{\parallel e} = 0$$

$\therefore$

$$\frac{\partial \tilde{n}}{\partial t} + \tilde{V}_r\frac{\partial n_0}{\partial r} + n_0\nabla_\parallel\Big(\underbrace{(\tilde{V}_{\parallel e}) - \tilde{V}_{\parallel i}}_{\underrightarrow{\tilde{J}_\parallel}\ (\text{model})} + \tilde{V}_{\parallel i}\Big) = 0$$
$$+\,\tilde{V}\cdot\nabla\tilde{n}$$

$$+\,\cancel{\tilde{V}_{\parallel i}}$$

neglect, acoustic coupling

So

$$\frac{\partial}{\partial t}\frac{\tilde{n}}{n_0} + \frac{\tilde{V}_r\nabla_r n_0}{n_0} + \frac{\underline{V}\cdot\nabla\tilde{n}}{n_0}$$
$$= +D_\parallel\nabla_\parallel^2\left(\phi - T\frac{\tilde{n}}{n_0}\right)$$

→ with D, ν:

$$\partial_t \nabla_\perp^2 \phi \rightarrow \partial_t \nabla_\perp^2 \phi - \nu \nabla^2 \nabla^2 \phi$$

$$\partial_t n \rightarrow \partial_t n - D \nabla^2 n$$

Reality: $D \ll \nu$.

Scale independent damping invoked frequently for zonal mode, i.e.

$$\partial_t \nabla_r^2 \phi_Z \rightarrow \partial_t \nabla_r^2 \phi_Z + \underset{\downarrow}{\mu} \nabla_r^2 \phi$$

i.e. magnetic inhomog. damping

Now: Important!

→ H–W coupled eqns. for n, ϕ

Coupling: ω vs. $D_\parallel \nabla_\parallel^2$

⇒ dimensionless parameter:

$k_\parallel^2 v_{The}^2 / \omega \nu_{ei}$ → adiabaticity parameter

→ α

So

$$\boxed{\frac{d\tilde{n}}{dt} + \frac{\tilde{v}_r}{n_0}\partial_r n_0(r) = D_\parallel \nabla_\parallel^2\left(\tilde{\phi} - \frac{T\tilde{n}}{n_0}\right)}$$

HW 2

So Finally, H-W equations:

$$\boxed{\begin{aligned} \rho_s^2 \frac{d}{dt}\nabla_\perp^2\tilde{\phi} &= D_\parallel \nabla_\parallel^2\left(\tilde{\phi} - \frac{T\tilde{n}}{|e|\,n}\right) \\ \frac{d\tilde{n}}{n_0\,dt} + \tilde{v}_r\frac{\partial_r\langle n_0\rangle}{n_0} &= D_\parallel \nabla_\parallel^2\left(\tilde{\phi} - \frac{T\tilde{n}}{|e|\,n_0}\right) \end{aligned}}$$

H-W Eqns.

Comments:

→ drift ⇒ drift acoustic, retain $\tilde{v}_\parallel$, in density eqn. + parallel ion momentum

→ obviously 2 field model

— dissipative coupling.

$\alpha \cong v_{The}^2 / R^2 q^2 \omega \nu_{ei}$ → α with

$k_\parallel = 1/Rq$

N. B. $k_\parallel$ not necessarily $1/Rq$!

{ sheared slab : $k_\parallel = k_\theta x / L_s$
{ Multi-scale

→ Why? → examine waves (c.f. crank out directly)

strongly coupled → adiabatic → drift mod. wave limit

$k_\parallel^2 D_\parallel / \omega > 1$

weakly coupled → hydrodynamic → convective cell limit

$k_\parallel^2 D_\parallel / \omega < 1$

N.B.: Each case → $\underline{\underline{2}}$ linear modes.

i.) Adiabatic Limit $\frac{k_\parallel^2 v_{The}^2}{\omega \nu} \geq 1$

[Fluid element diffuses $\lambda_\parallel$ faster than 1 oscillation

[distinct from lmfp vs $\lambda_\parallel$!

Now $\frac{k_\parallel^2 v_{The}^3}{\omega \nu} > 1$

$$\frac{\tilde{n}}{n} \simeq \frac{|e| \tilde{\phi}}{T} + \tilde{h}$$

↑
l.o. Boltzmann

Plugging in:

$$\frac{\partial \tilde{h}_{\underline{k}}}{\partial t} + k_\parallel^2 D_\parallel \tilde{h}_{\underline{k}} = -\frac{|e|}{T}\partial_t \tilde{\phi} - v_* \partial_y \frac{|e|\tilde{\phi}}{T}$$

$$\rho_s^2 \frac{\partial}{\partial t} \nabla_\perp^2 \tilde{\phi} = + k_\parallel^2 D_\parallel \tilde{h}_{\underline{k}}$$

so

$$\tilde{h}_{\underline{k}} = +i\frac{|e|}{T}\frac{(\omega-\omega_*)\tilde{\phi}}{\cancel{-i\omega} + k_\parallel^2 D_\parallel}$$

$$\cong i\frac{|e|}{T}\tilde{\phi}\,\frac{(\omega-\omega_*)}{k_\parallel^2 D_\parallel}$$

so

$$\langle \tilde{v}_r \tilde{n} \rangle = \cancel{\left\langle \tilde{v}_r \frac{|e|\tilde{\phi}}{T} \right\rangle} + \langle \tilde{v}_r \tilde{h} \rangle$$

$$\cong \sum_{\underline{k}} \rho_s c_s \left(\frac{|e|\tilde{\phi}_{\underline{k}}}{T}\right)^2 \frac{k_\theta(\omega-\omega_*)}{\underset{\uparrow}{k_\parallel^2 D_\parallel}}$$

$$\neq 0$$

$\Rightarrow$ Parallel dissipation induces ϕ, n phase shift $\rightarrow$ relaxation.

And note:

$\langle \tilde{v}_r \tilde{n} \rangle > 0 \Rightarrow \omega < \omega_{*e}$

What is ω_{real}?

$k_\parallel^2 v_{Th}^2 / \nu_{ei}\, \omega \gg 1 \Rightarrow$

$\frac{\tilde{n}}{n} \cong \frac{|e|\tilde{\phi}}{T}$ and

$$\rho_s^2 \frac{d}{dt} \nabla_\perp^2 \tilde{\phi} \simeq \frac{1}{n_0}\left(\frac{d\tilde{n}}{dt} + \tilde{v}_r\, \partial_r \langle n_0 \rangle\right)$$

$\Rightarrow$

$$\partial_t \frac{|e|\tilde{\phi}}{T} - \rho_s^2 \frac{d}{dt} \nabla_\perp^2 \tilde{\phi} + v_*\, \partial_y \frac{|e|\tilde{\phi}}{T} = 0$$

i.e. For adiabatic limit;

HW → HM!

[N.B: HW contains HM

So $\boxed{\omega \cong \omega_* / 1 + k_\perp^2 \rho_s^2}$

N.B.

Other mode? ⇒ $O(1/\alpha) \rightarrow 0$

heavily damped.

so

$$\langle \tilde{v}_r \tilde{n} \rangle = \underbrace{\sum_{\underline{k}} + \rho_s c_s \left| \frac{e\hat{\phi}_k}{T} \right|^2 \frac{k_\theta k_\perp^2 \rho_s^2 \omega_*}{k_\parallel^2 D_\parallel (1 + k_\perp^2 \rho_s^2)}}_{>0}$$

$$\boxed{\text{N.B.: } \left\{ \begin{array}{l} \text{Relaxation} \\ \text{Growth} \end{array} \right. \Rightarrow \begin{array}{l} \text{Parallel friction/} \\ \text{diffusion.} \\ \underline{\text{and}} \\ \omega < \omega_* \end{array}}$$

Physics: $\omega < \omega_*$ ⇒ 'gain' from gradient relaxation exceeds penalty/cost of "pumping"

For growth rate:

From HM:

$$\frac{\omega_*}{\omega} - 1 - k_\perp^2 \rho_s^2 = 0 \qquad \underline{\text{D.R.}}$$

$$\overbrace{\underset{\substack{\uparrow \\ \text{diamagnetic}}}{\frac{\omega_*}{\omega}} - \underset{\substack{\uparrow \\ \text{pol.}}}{k_\perp^2 \rho_s^2}}^{\text{ions}} = \underset{\substack{\uparrow \\ \text{Boltzmann}}}{\overset{\underline{\text{electrons}}}{1}}$$

now $\frac{\tilde{n}}{n} = \frac{|e|\tilde{\phi}}{T}\left[1\right] + \underbrace{\tilde{h}}_{O(1/\alpha)}$

Treat via perturbation theory:

$$\frac{\omega_*}{\omega} - k_\perp^2\rho_s^2 = 1 + \overbrace{i\frac{(\omega-\omega_*)}{k_\parallel^2 D_\parallel}}^{i\delta}$$

l.o. $\frac{\omega_*}{\omega} - k_\perp^2\rho_s^2 = 1$ ✓

1st O. $-\omega_*\frac{\delta\omega}{\omega^2} \cong i\left(\frac{\omega_r-\omega_*}{k_\parallel^2 D_\parallel}\right)$

$$\frac{\delta\omega}{\omega} = i\frac{(\omega_* - \omega_r)}{k_\parallel^2 D_\parallel(1+k_\perp^2\rho_s^2)}$$

$$= \frac{i\,\omega_* k_\perp^2\rho_s^2}{k_\parallel^2 D_\parallel(1+k_\perp^2\rho_s^2)^2}$$

Drift wave instability!

$\Big\{$ Collisional / Dissipative / Resistive

Drift Wave Instability.

→ All electron drift instabilities TEM are, in some sense, similar.

$ik_\parallel^2 D_\parallel$ → CDW

$\omega \sim k_\parallel v_\parallel$ → collisionless DW
(resonance → Landau)

CHW!) solve via DKE.

$i\nu_{ee_{\text{eff}}}$ → DTEM

$\omega \sim \omega_{De} E$ → CTEM.
(resonance) → precession)

N.B. $\left\{ \begin{array}{l} \nabla T_e \text{ enters, too} \\ \text{can vary} \end{array} \right.$

HW - Work out collisionless drift wave with ∇n, ∇T.

ii.) Hydrodynamic Limit

$k_\parallel^2 D_\parallel / \omega < 1$ ⇒ oscillates faster than collisional parallel diffusion }

→ Electrons don't 'Boltzmann-ize'!

→ Akin MHD

⇒ convective cells!

$\omega_r, \omega_{Ion} \sim \sqrt{\alpha}$ → low

Generally unimportant, except high density, cool edge

⇒ revisit for density limit!

N.B.

- Essential to crank thru yourself.
- Re Ohm's Law → key!

$$E_\parallel + \nabla_\parallel P_e = \eta \underset{\textcircled{4}}{J_\parallel}$$

$$\underset{\textcircled{1}}{-\frac{1}{c}\frac{\partial A_\parallel}{\partial t}} \quad \underset{\textcircled{2}}{-\nabla_\parallel \phi} \qquad\qquad \underset{\textcircled{3}}{\nabla_\parallel n}$$

So ① ~ ② → ideal MHD

$E_\parallel \sim 0$

② ~ ④ → resistive MHD

with ① ($\alpha < 1$)

② ~ ③ → drift wave

with ④ ($\alpha > 1$)

Structure of Ohm's Law largely determines Balances } Dynamics.

and Electron Inertia + $E_\parallel$ → ETG

EMHD.

→ Mean Field / Zonal Flows

Can zonally (θ, ϕ) avg H-W Eqns:

$$\partial_t \langle n \rangle + \partial_r \langle \tilde{v}_r \tilde{n} \rangle = S_n + D\nabla_r^2 \langle n \rangle$$

$$\wr$$

$$\partial_r \langle \tilde{v}_r \tilde{h} \rangle$$

→ expression previous is QL flux calculation

→ ($\sim$) good, dissipative dynamics

Concern: NL frequency shift.

But, *equally* have:

$$\partial_t \langle \underset{\Downarrow}{\nabla_r^2 \phi} \rangle + \overset{\Downarrow}{\partial_r \langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle} = \mu \nabla_r^2 \langle \nabla_r^2 \phi \rangle$$

polarization charge.

- Zonal Flow evolution clear.
- *Key* → vorticity Flux
- equal footing with mean field density evolution

N.B.: If a particle flux, then likely also a vorticity flux
⇒ Zonal flow evolution.

2 Questions

→ physics of vorticity Flux?

→ Relation between particle and vorticity flux ⇒ Zonal flow generation.

→ Vorticity Flux

$$\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle = \langle \partial_y \tilde{\phi} (\partial_x^2 \tilde{\phi} + \partial_y^2 \tilde{\phi}) \rangle$$

$$= \langle \partial_y \tilde{\phi} \, \partial_x^2 \tilde{\phi} \rangle$$

i.e. $i k_y k_y^2 \rightarrow$ odd k_y $\quad \langle \rangle = \sum_{\underline{k}}$

$$\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle = \langle \partial_x (\partial_y \tilde{\phi} \, \partial_x \tilde{\phi}) - \underbrace{\cancel{(\partial_{y,x} \tilde{\phi} \, \partial_x \tilde{\phi})}}_{\text{odd } k_y} \rangle$$

$$= \langle \partial_x (\partial_y \tilde{\phi} \, \partial_x \tilde{\phi}) \rangle$$

$$= \partial_x \underbrace{\left(\langle \partial_y \tilde{\phi} \, \partial_x \tilde{\phi} \rangle \right)}_{\Downarrow}$$

so

Reynolds stress !
(E×B)

Vort. Flux = Reynolds Force (E×B)

i.e. Vort Flux drives E×B Flow.

N.B.: → 1 direction of symmetry utilized

→ McIntyre and R. Wood Theorem

⇒ PV mixing + 1 direction of symmetry

⇒ Zonal flow formation!

→ Welcome to Taylor Identity!
(G. I. Taylor, 1915)

→ HW: ① $\langle \tilde{B}_r \tilde{J}_\parallel \rangle = ?$
(Magnetic Taylor Identity)

② Relate to Polarization charge balance.

→ Relation?

Back to PV, ...

Inviscid HW Eqn:

$$\rho_s^2 \frac{d}{dt} \nabla_\perp^2 \phi = D_\parallel \nabla_\parallel^2 \left(\tilde{\phi} - \frac{T\tilde{n}}{|e| n_0} \right)$$

$$\frac{1}{n_0} \frac{d}{dt} \tilde{n} + \tilde{v}_r \partial_r \langle n \rangle = D_\parallel \nabla_\parallel^2 \left(\phi - \frac{T \tilde{n}}{|e| n_0} \right)$$

$\delta n \equiv \tilde{n} / n_0$

Then, subtract:

$$\frac{d}{dt}\underbrace{\left(\hat{n} - \rho_s^2\nabla^2\hat{\phi}\right)} + \tilde{V}_r\,\partial_r\left(\frac{\langle n\rangle}{n_0} - \rho_s^2\frac{\langle\nabla_r^2\phi\rangle}{n_0}\right)$$

$$= 0 \qquad \longrightarrow \qquad \underline{PV} \rightarrow \begin{array}{c}\text{total charge}\\ \text{GC, polarization}\end{array}$$

i.e.

$$\frac{d}{dt}\tilde{q} + \tilde{V}_r\,\partial_r\langle q\rangle = 0$$

$$\langle\tilde{q}^2/2\rangle \equiv \underline{\text{Potential Enstrophy}}$$

~~production~~

and

$$\partial_t\frac{\langle\tilde{q}^2\rangle}{2} + \partial_r\frac{\langle\tilde{V}_r\tilde{q}^2\rangle}{2} + \langle\tilde{V}_r\tilde{q}\rangle\,\partial_r\langle q\rangle = 0$$

⇓ ⇓

spreading (transport pot Enstrophy) PV flux

then:

$Pr \sim 1.$

$$\left[\partial_t\frac{\langle\tilde{q}^2\rangle}{2} + \partial_r\frac{\langle\tilde{V}_r\tilde{q}^2\rangle}{2}\right]\Big/\,\partial_r\langle q\rangle + \cancel{\frac{\nu\langle(\nabla\tilde{q})^2\rangle}{\partial_r\langle q\rangle}}$$

⇓

$$= -\langle\tilde{V}_r\hat{n}\rangle + \langle\tilde{V}_r\rho_s^2\nabla^2\hat{\phi}\rangle$$

$$\partial_t\langle V_E\rangle + \mu\langle V_E\rangle = -\partial_r\langle\tilde{V}_{r_E}\tilde{V}_{\theta_E}\rangle$$

$$= -\langle\tilde{V}_r\rho_s^2\nabla^2\tilde{\phi}\rangle$$

$$\partial_t \left[\langle V_E \rangle + \frac{\langle \tilde{Z}^2 \rangle}{2} \Big/ \partial_r \langle q \rangle \right] + \partial_r \left[\frac{\langle \tilde{v}_r \tilde{Z}^2 \rangle}{2} \Big/ \partial_r \langle Z \rangle \right] + \mu \langle V_E \rangle + \text{visc.} = - \langle \tilde{v}_r \tilde{n} \rangle$$

~ variant of Charney - Drazin Thm.

~ 3 messages:

ⓐ $\langle V_E \rangle$ locked to $\underset{\underset{\text{WMD}}{\wr}}{\langle \tilde{Z}^2 \rangle / 2 \partial_r \langle Z \rangle}$

up to damping, spreading, turbulent particle flux.

ⓑ in steady state,

$\langle \tilde{v}_r \tilde{n} \rangle$ locked to PE damping and spreading.

ⓒ in steady state:

$$\underbrace{\langle \tilde{v}_r \tilde{n} \rangle \simeq \langle \tilde{v}_r \rho_s^2 \nabla_\perp^2 \tilde{\phi} \rangle}_{} + \underbrace{\underbrace{\text{Visc} + \text{spreading}}_{}}_{}$$

relation of flux
and E×B Reynolds force.

Must evolve ZF and mean density on equal footing.

→ 2 types of structure with zonal symmetry

- Zonal flow
- corrugation