Physics 218c

Lecture 9a.: – TEM and TIM, conclusion.

– Geometry ⟺ Dynamics I.

– OV

– Magnetic Shear and Quasimodes / Twisted Slices

– Shearing Coordinates

→ **Geometry and Dynamics**

- Till now, homogeneous media
- Tokamak – non-trivial geometry, which enters physics
  ⇒ compatibility issues.
  – here consider basic elements
- Basic Elements – [Magnetic, $E \times B$ on equal footing

  - Resonances / Resonant Surfaces
    → spatial
    → wave-particle

  - Shear ↔ Representation / Quasimodes Shearing coordinates
    → Magnetic
    → $\underline{E \times B}$
    shearing coordinates

  - Toroidicity ⇒ "Ballooning"
    ⇒ Representing Bloch Modes
    poloidal harmonic coupling (real space)

⇒ [redacted] "Ballooning Mode Formalism" (along line)

Some reading:

- Roberts + Taylor '65 — shearing physics, Quasimodes
[* — good for intuition]
- CHT — Ballooning formalism
- Goldreich + Lynden-Bell (galactic dynamics) — { shearing coordinates
- BDT
Rhines + Young 〉 Diffusion/scattering + shear.

N.B. With toroidicity ⇔ trapped particles.

Extended OV

(a)

- Resonances.

→ demand poloidal, toroidal periodicity

$$\hat{\phi} \sim \hat{\phi}_{m,n}(r)\, e^{i(m\theta - n\phi)}$$

{ $\frac{m}{n}$ mode pitch

→ Field lines wind ⟺ pitch

$$\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{R\,d\phi}{B_z}$$

$$\frac{r\,d\theta}{R\,d\phi} = \frac{B_\theta}{B_z} \,, \qquad \theta = \frac{\phi}{q(r)} + \theta_0$$

$q(r) =$ [redacted] $\cdot \dfrac{r\,B_z}{R\,B_\theta(r)}$ ≡ defines pitch of magnetic [redacted] field lines

so

→ when $q(r) = \dfrac{m}{n}$ ⇒ [mode pitch = field line pitch]

⇒ resonant surfaces.

where

$$(B \cdot \nabla)\tilde{\Phi} = 0$$

i.e.

$$\left(B_r \partial_r + \frac{B_\theta}{r}\frac{\partial}{\partial\theta} + \frac{B_z}{R}\frac{\partial}{\partial\phi} \right) \tilde{\Phi} = 0$$

i.e. $\left(i \frac{m}{r} B_\theta - i \frac{n}{R} B_z \right) \Phi = 0$

$\Rightarrow$

$$k_\parallel = \frac{\underline{k} \cdot \underline{B}_0}{|B_0|} = \frac{m}{q(r)} - n$$

rational surfaces have $k_\parallel = 0$

i.e. minimize:

— Landau damping

$$\left[\frac{1}{\omega - k_\parallel v_\parallel} \right]$$

$\hookrightarrow$ magnetic resonance $\Leftrightarrow$ wave particle resonance

— line bending

$$\delta W = \int d^3x \left[\frac{(\nabla \times \xi \times B_0)^2}{4\pi} + \cdots \right]$$

+ definitely stabilizing

$$(B_0 \cdot \nabla \xi)^2 \sim k_\parallel^2 B_0^2 \xi^2$$

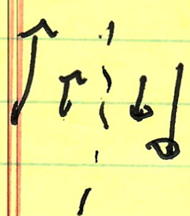
$\Rightarrow$ modes related to / situated at resonant surfaces.

→ $\underline{k}\cdot\underline{B} = 0$ resonances <u>critical</u>

to tokamaks

→ sites of modes, islands, etc.

$\otimes B_0$



locally → transform to slab

⇒ tearing, reconnection

→ Low q resonances:

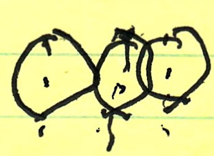
– tearing, NTM, etc.

⇒

MHD → disruptions

→ resonances coupled by toroidicity

→ resonance overlap → $\left\{ \begin{array}{l} \text{chaos} \\ \text{stochasticity} \end{array} \right.$



w_I vs Δr.

$\Rightarrow$ N.B. $\underline{B} \cdot \underline{\nabla}$ operator is important

$\rightarrow \delta W_v$ etc. $\Leftrightarrow$ $\left\{ \begin{array}{l} \text{Shear} \\ \text{Alfven} \end{array} \right.$

$\rightarrow$ kinetics $v_{\parallel} \hat{n} \cdot \nabla$ $\qquad \hat{n} = \frac{\underline{B}}{|B|}$

Note can write:

$$B \cdot \nabla = \frac{B_0}{R} \frac{\partial}{\partial \phi} + \frac{B_\theta}{r} \frac{\partial}{\partial \theta} + \underline{\tilde{B}} \cdot \underline{\nabla}$$

$R \partial \phi = \partial z$

$$= B_0 \partial_z + \frac{B_\theta}{r} \frac{\partial}{\partial \theta} + \underline{\tilde{B}} \cdot \underline{\nabla}$$

$$= \partial_z + \frac{1}{R q(r)} \frac{\partial}{\partial \theta} + \underline{\tilde{B}} \cdot \underline{\nabla}$$

analogous to:

$$\frac{d}{dt} = \partial_t + \langle v_y \rangle \partial_y + \underline{\tilde{v}} \cdot \underline{\nabla}$$

$$= \partial_t + \langle v_y(x) \rangle \partial_y + \tilde{v} \cdot \nabla$$

so $\quad \underline{B} \cdot \underline{\nabla} \to \partial_z + \frac{1}{R q(r)} \frac{\partial}{\partial \theta} + \underline{\tilde{B}} \cdot \underline{\nabla}$

$\quad\quad q \to$ sheared $=$ variable winding rate

$\frac{d}{dt} \to \partial_t + \langle v_y(x) \rangle \partial_y + \underline{\tilde{v}} \cdot \underline{\nabla}$

Also:

$$\frac{1}{\left[\underbrace{\omega - k_y \langle v_y(x) \rangle} - \cancel{k_\parallel v_{th}}\right]}$$

sheared flow $\Leftrightarrow$ Landau resonance.

this brings us to:

(b) Shear

Shear $\equiv$ variable winding / flow rate

$\to$ excitations twist

$\Rightarrow$ affects coherence of fluctuations in space / time

$\Rightarrow$ ~ fluctuation 'wants' to align with shear, but can't.

$\Rightarrow$ shearing coordinates!

- a natural way to describe fluctuations in shear

- but price $\rightarrow$ lose normal mode description
  $\rightarrow$ shearing "quasi-modes"

Care? $\rightarrow$ does shearing quasi-mode grow to NL prior to decay, etc.?

Shearing coordinates:

$$\big(\underbrace{\partial_t + V_y' x\, \partial_y}_{\text{seek eliminate shearing}} + \tilde{\underline{V}}\cdot\underline{\nabla}\big)\, c = 0$$

so

~ define new coordinates, co-moving with $\langle V_y \rangle$ flow

→ i.e. tilting coordinates

$$x' = x$$

$$y' = y - V' x t \qquad \text{(simplify notation)}$$

$$z' = z$$

$$t' = t$$

and

$$\partial_x = \text{[redacted]} \; \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}$$

$$= \frac{\partial}{\partial x'} - V' t' \frac{\partial}{\partial y'}$$

$$\partial_y = \frac{dy'}{dy}\frac{\partial}{\partial y'} + \cancel{\frac{dx'}{dy}\frac{\partial}{\partial x'}} = \frac{\partial}{\partial y'}$$

$$\partial_t = \text{[redacted]} \; \frac{\partial t'}{\partial t}\partial_{t'} + \frac{\partial y'}{\partial t}\partial_{y'}$$

$$= \partial_{t'} - V' x' \partial_{y'}$$

So, in shearing coordinates:

$$\partial_t + V_y' x \, \partial_y = \partial_{t'} - V' x \cancel{\partial_{y'}} + V' x \cancel{\partial_{y'}}$$

$$= \partial_{t'}$$

i.e. coordinates eliminate fast variation ∂

$$c_k e^{i\underline{k}\cdot\underline{x}} \rightarrow \exp\left[i(k_x' - k_y' V' t')x + i k_y' y' + i k_z' z' \right]$$

$$= \exp\left[i(\underline{k}'\cdot\underline{x}') - i k_y' V' x' t \right]$$

$$= e^{i\underline{k}'\cdot\underline{x}'} e^{-i k_y' V' x t} = \exp\left[i x (k_x' - k_y' V t) + k_y' y' + k_z' z' \right]$$

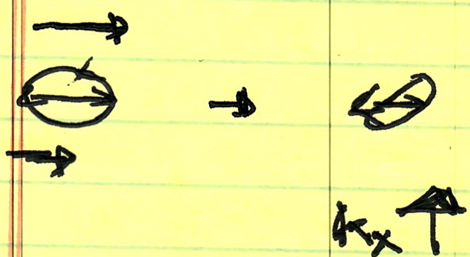
$t' = t$

i.e. $k_x \rightarrow k_x - k_y V' t$

as in: $\dfrac{d\underline{k}}{dt} = -\dfrac{\partial}{\partial \underline{x}}(\omega + \underline{k}\cdot\underline{V})$

Physics of shearing coordinates:

$\boxed{\text{Eddy tilting and thinning}}$:



$$k_x \to k_x - k_y V' t$$

shearing coordinates follow the tilt of the eddy

For diffusion/advection:

$$\left[\partial_t + V_y' x \, \partial_y - D\left(\partial_x^2 + \partial_y^2\right)\right] C = 0$$

$$C = C_0 e^{i \underline{k} \cdot \underline{x}} ,$$

$$\underbrace{[\quad] C}_{=0} = \left[\partial_{t'} + D\left[\left(k_x' - k_y' V' t\right)^2 + k_y'^2\right]\right] C$$

$$\partial_x = \partial_x' - V' t \, \partial_y'$$

so

$$C = C_0 e^{i(k_y' y + k_z z)} \times e^{-k_y'^2 D t} \exp\left[-\int dt \, D \left(k_x' - k_y' V' t\right)^2\right]$$

$$\sim e^{-k_y'^2 V'^2 t^3 D/3} \simeq \exp\left[-k_y'^2 D \frac{V'^2}{3} t^3\right]$$

shear enhanced decorrelation decay.

so have:

$$x' = x$$

$$y' = y - \frac{x}{L_s} z$$

$$z' = z$$

"twisted slicing coordinates"

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} - \frac{z'}{L_s} \frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$$

$$\partial_z = \partial_{z'} - \frac{x'}{L_s} \partial_{y'}$$

and again find:

$$\frac{\partial}{\partial z} + \frac{x}{L_s} \frac{\partial}{\partial y} = \partial_{z'} \cancel{- \frac{x'}{L_s} \partial_{y'}} \cancel{+ \frac{x}{L_s} \frac{\partial}{\partial y}} = \partial_{z'}$$

$\Rightarrow$

Twisted slicing coordinates annihilate leading, fast behavior $\nabla_{\parallel}$. $\Rightarrow$ natural eikonal variables.

i.e. eliminates fast variation $\underline{B}\cdot\underline{\nabla}$ due shear.

$$\nabla_{\parallel}\bar{\underline{\Phi}} = 0$$

$$\underline{\Phi}_k e^{i\underline{k}\cdot\underline{x}} \rightarrow \bar{\underline{\Phi}}^{(0)} \exp\left[i\left(k_x' x' - k_y' \frac{x'}{L_s} z' x'\right)\right]$$

$$e^{ik_y' y'}$$

$$= \exp\left[i(\underline{k}'\cdot\underline{x}')\right] e^{-ik_y' \frac{x'}{L_s} z}$$

i.e. $k_x \rightarrow k_x - k_y \dfrac{z}{L_s}$

as in:

$$B_y = \frac{B_0}{L_s} x$$

$$\boxed{\frac{dk_x}{dz} = -\frac{\partial}{\partial x}\left(k_y' \frac{B_y}{B_0}\right)}$$

$\rightarrow$ can apply to zonal fields

– effective eikonal, evolving on z

→ What of "magnetic shearing coordinates"?

⇒ Twisted Slicing Modes

As before:

$$\nabla_{\parallel} = \frac{k \cdot B_0}{|B_0|} \rightarrow \frac{\partial}{\partial z} + \frac{1}{Rq}\frac{\partial}{\partial \theta}$$

$$\frac{1}{Rq} \simeq \frac{1}{\left[Rq(r_0+x)\right]} \simeq \frac{r}{Rq(r_0)} - \frac{x\, q' r}{Rq^2}\frac{\partial}{r\partial\theta}$$

$$\cong \left(\frac{r}{Rq(r_0)} + \frac{x}{L_s}\right)\frac{\partial}{\partial y}$$

↳ absorb to ∂z

$$1/L_s = \frac{-r q'}{q R q} \equiv \frac{-\hat{s}}{Rq}$$

$$\nabla_{\parallel} = \frac{\partial}{\partial z} + \frac{x}{L_s}\frac{\partial}{\partial y}$$

then use shearing coordinates with:

$$t \rightarrow z\,, \qquad \frac{1}{L_s} \rightarrow V_y'$$

$$k_x \to k_x - \frac{k_y}{L_s} z \quad , \quad k_y \to k_y$$
$$k_z \to k_z$$

Andy with diffusion: (general scattering)

$$\left[\frac{\partial}{\partial z} + \frac{x}{L_s} \frac{\partial}{\partial y} - \underset{\substack{\uparrow}}{D} \left(\partial_x^2 + \partial_y^2 \right) \right] f = 0$$

i.e. $D_M = \sum_k |b_k|^2 \pi \delta(k_\parallel)$

i.e. $v_\parallel \hat{n} \cdot \underline{\nabla} f - |v_\parallel| D \nabla^2 f = 0$

$$[\quad] f = \left[\partial_{z'} + D \left[\left(k_x' - k_y \frac{z}{L_s} \right)^2 + k_y'^2 \right] \right] f$$

$$f_{\underline{k}} = f_{\underline{k}} \, e^{-k_y'^2 D z} \exp\left[- \int dz \, D \left(k_x' - k_y' \frac{z}{L_s} \right)^2 \right]$$

$$\sim e^{-\frac{k_y'^2}{3} \frac{D}{L_s^2} z^3} \cong \exp\left[- \frac{k_y'^2}{3} \frac{D}{L_s^2} z^3 \right]$$

magnetic shear induced decorrelation $\Big\} \to$ line correlation length

if retain V_{Te}:

$$1/\tau_c = \left(\frac{k_y^2 V_{the} D}{3 L_s^2} \right)^{1/3} \leftarrow \text{time}$$

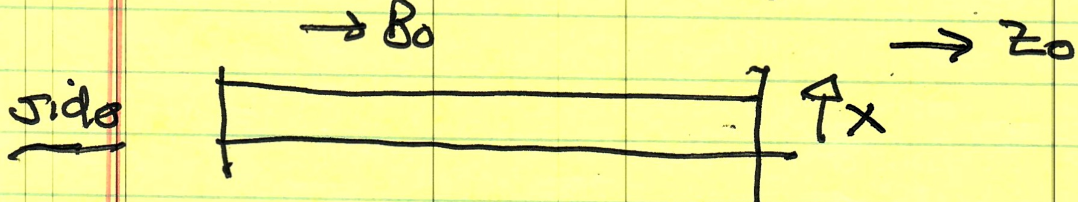
corresponding

→ What does it Mean?

— if, say, consider interchange,

$$\gamma^2 = \frac{k_y^2}{k_\perp^2}\,\frac{g}{L_p}$$

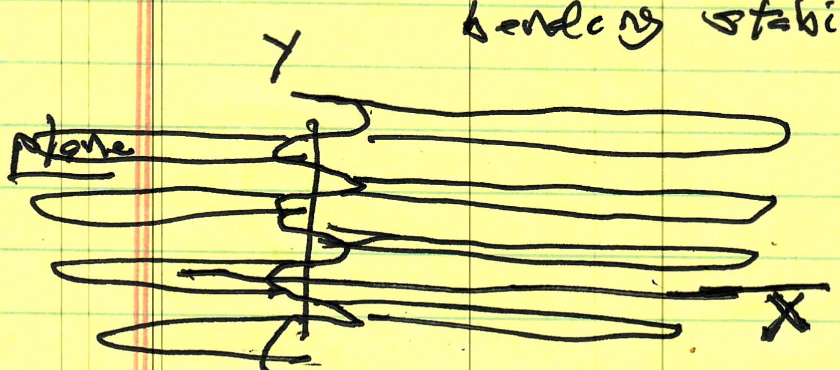
— if B = const and line-tying (i.e. conducting B.C.)



need keep $k_\parallel$ down ~ k_z

$$\gamma^2 = \frac{k_y^2}{k_\perp^2}\,\frac{g}{L_p} - V_A^2 k_\parallel^2$$

bending stabilization



slices (à la' loaf of bread)

↳ ripple on y (à la' flute)

so slices extend in x, but high k_y.

Now, if $B = B_0\left(\hat{z} + \frac{x}{L_s}\hat{y}\right)$

shear

slices must align to remain aligned with field, locally.

i.e. $k_z \to k_z' - \frac{x}{L_s} k_y'$

$$\partial_z \to 0 \Rightarrow \frac{\partial}{\partial z'} - \frac{x'}{L_s}\partial_y = 0$$

maintain alignment

$$\frac{dy'}{dz'} = \frac{x'}{L_s}$$

mode/cell must twist in z

$\Rightarrow$ $\boxed{\text{twisted slices}}$

$\rightarrow$ analogue of eddy tilting for shear flow.

...n in the... ...and also...
...nation) and also ...
...III and IV we deal w... while the principal ap...
...in Secs. V and VI is ...
...which can be shown
...discussed

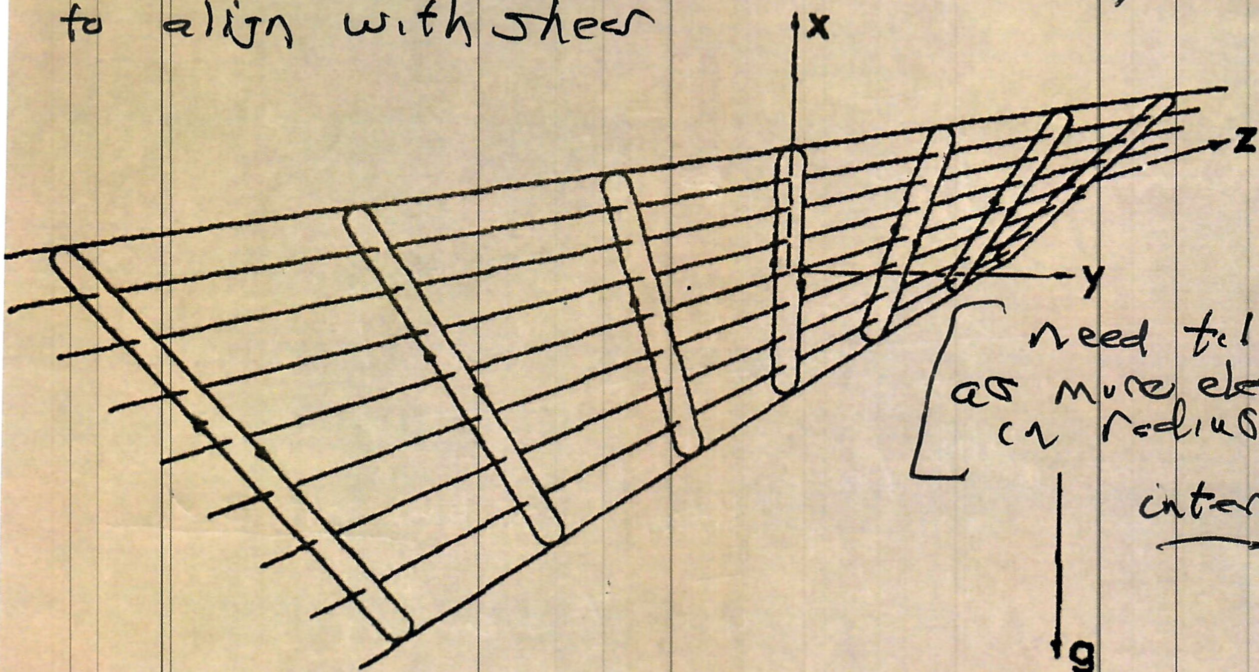
$X = 0$

$B = B_{0}z$

$\underline{B} = B_{0z}\left(\hat{z} + \frac{x}{L_s}\hat{y}\right)$

Fluctuation twists to align with shear

↗ z



need tilt as more element in radius

interchange

Fig. 1. Twisted slicing mode.

...red magnetic field through a series of simpler ...related problems. In Sec. III we first consider the ...itational instability of a perfectly-conducting ...mpressible fluid in a sheared magnetic field, ...ained between conducting endplates which, how- ... are coated with a thin insulating layer so that

Which brings us to:

→ Resistive Interchanges!

Recall:

- interchange ⟺ $k_\parallel = 0$
  (flute)

- stabilized by shear
  $k_\parallel = k_y x / L_s$ $\quad (\hat{E}_\parallel = 0)$

- then introduce resistivity →
  decouple field and fluid;
  x scale and $k_\parallel$ linked.

$$\nabla_\perp^2 \hat{\phi} + \frac{V_A^2}{\gamma \eta} \nabla_\parallel^2 \hat{\phi} + \frac{g}{|L_\rho| \gamma^2} k_y^2 \hat{\phi} = 0$$

$\nabla_\parallel = i k_\parallel(x)$

$\gamma \sim O(\eta^{1/3}) \rightarrow O(1/S^{1/3})$

$1/S = a\eta / a^2 V_A$

spatial width $\sim \mathcal{O}(a/s^{1/3})$.

$w \sim a/s^{1/3}$

$s \gg 1$.

Now, consider twisting coordinates, clarity

assume infinite length. (no b.c.)

$x = y - \frac{x}{L_s} z \qquad \varphi = z$

$x \to y'$ ... $\varphi \to z'$

$\xi = x$

$\xi \to x'$

z', φ assume to extend $\pm\infty$, along line. Does not satisfy boundary condition. Quasi-mode packet

following formalism reconciles twisted slice and periodicity

and $\frac{\partial}{\partial y} \Rightarrow \frac{\partial}{\partial x}$

$$\frac{\partial}{\partial x} \Rightarrow \frac{\partial}{\partial \xi} - \frac{\varphi}{L_s}\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial z} \Rightarrow \frac{\partial}{\partial \varphi}$$

So

$$\boxed{\left[\left(\frac{\partial}{\partial \xi} - \frac{\varphi}{L_s}\right)^2 + \frac{\partial^2}{\partial x^2}\right]\hat{\phi} + \frac{v_A^2}{\gamma\Omega}\frac{\partial^2 \hat{\phi}}{\partial \varphi^2} - \frac{g}{|L_\rho|}\frac{\partial^2}{\partial x^2}\hat{\phi} = 0}$$

Lin Eqn in twisted slicing coordinates:

line, radius, binormal

$$\hat{\phi} = \hat{\phi}(\varphi)\, e^{i k_x \xi}\, e^{i k_y \chi}$$

$$= e^{i k_x x}\, e^{+ i k_y (y - \frac{x}{L_s} z)}\, \hat{\phi}(z)$$

'eigenfunction' describes length along field lines

opens door to "ballooning"

$\Rightarrow$

$$\frac{\partial^2 \hat{\phi}}{\partial \varphi^2} - \gamma \frac{\Omega k_y^2}{v_A^2}\left[\left(\frac{k_x}{k_y} - \frac{\varphi}{L_s}\right)^2 + 1\right]\hat{\phi}$$

$$+ \frac{k_y^2 g}{L_\rho}\frac{\gamma\Omega}{v_A^2 \gamma^2}\hat{\phi} = 0$$

$k_x \ll k_y$ (large scale)

differs from usual resistive interchange

$$\frac{\partial^2 \tilde{\phi}}{\partial \varphi^2} - \gamma \frac{\eta k_y^2}{V_A^2}\left[\frac{\varphi^2}{L_S^2} + 1\right]\tilde{\phi} + \frac{\eta k_y^2 \gamma_I^2}{\gamma V_A^2}\tilde{\phi} = 0$$

$$\tilde{\phi}(\varphi) = e^{-\alpha \varphi^2/2}$$

$\Rightarrow$

$\gamma \tau_A \sim S^{-1/3}$

$\gamma \sim \eta^{1/3}$

$\sqrt{\alpha} \sim \frac{1}{L_{eff}} \sim \eta^{1/3}$

$L_{eff} \sim \eta^{-1/3}$

effective length along field line

What is new?

$\Rightarrow$ $k_x a \sim 1$ i.e. no constraint on spatial scale.

$k_x \ll k_y$

$\Rightarrow$ like slice

Comments

- $\vec{\phi} = f(x)\, \vec{\phi}(z)\, e^{i k_y (y - \frac{x}{L_s} z)}$

  $f(x)$: envelope in x, slow

  $\vec{\phi}(z)$: eigenfunction, $l_{\parallel} \sim \eta^{-1/3}$

  $e^{i k_y (y - \frac{x}{L_s} z)}$: twisting poloidal variation

- here "modes" are finite length rolls, locally aligned with shear

  ϕ const along $y - \frac{z\,x}{L_s}$ = const.

- "modes" here are really $\boxed{\text{wave-packets}}$ of localized resistive interchange

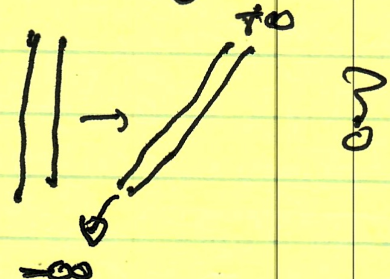
  "quasimodes"

eigenmodes

"modes", "quasi-modes" → no b.c. in z, φ

modes → satisfy b.c. (periodicity) in ϕ.

- "quasimodes" have finite length in
  z → allow twist to align with shear
  ⇒ energetics!



- $\gamma \sim \eta^{1/3}$ but $k_x a \sim 1$
  → large transport

contrast: $\gamma \sim \eta^{1/3}$, $\Delta \sim \eta^{1/3}$

$$-\eta \frac{\partial p_0}{\partial r} = \Gamma_p$$

↓

<u>weak.</u>

- wave packets disperse, but relevant.

Poloidal coupling due toroidicity

⇒ wave packets become eigenmodes

TBC