Physics 218 C

Lecture 5b → Geometry ⟺ Dynamics II

- review - Magnetic Shearing
- Resistive Interchange
- Toroidicity
- Ballooning Representation

Recall shear: velocity / magnetic

Magnetic shear:

$$- \nabla_\parallel = \frac{\partial}{\partial z} + \frac{1}{Rq(r)}\frac{\partial}{\partial \theta}$$

↳ pitch of field lines varies with radius.

$$1/Rq = \frac{1}{Rq(r_0)} + \frac{x}{L_s}$$

↳ shear

$$1/L_s = -\hat{s}/Rq$$

→ Plan:

M → Geometry 2 ✓

W → Shearing and Zonal Modes
⇒ Saturation

M → L→H and ITB

W → Rotation and Momentum Transport

M → Holiday

W → Density Limit and Greenwald Scaling, SOL

M (Finals Week) → Tynan on PWI.

→ After Week 8, theoretical treatment less intensive

→ Fall 218b ⇔ Tokamak MHD emphasis
(available online)
(Different previous)

so shearing coordinates, with

$f \to z$

N.B.: z no longer periodic, corresponds to distance along line

$V_y' \to 1/L_s$

$k_x \to k_x - k_y z/L_s$, $k_y \to k_y$, $k_z \to k_z$

on

$$\frac{dk_x}{dz} = -\frac{k_y}{L_s} \iff \frac{dk_x}{dt} = -k_y V_y'$$

and

$$1/\ell_{\parallel c} \sim \left(k_y^2 D/3L_s^2\right)^{1/3}$$ → decorrelation length

$D \sim D_M$

and $k_z \to k_z' - \frac{x}{L_s} k_y'$

$$\partial/\partial z' - x'/L_s\, \partial/\partial y' = 0 \Rightarrow$$

$$\frac{dy'}{dz} = \frac{x'}{L_s}$$ mode/cell must twist in z

→ analogue of eddy tilt.

N.B. ; Zonal structures:

Velocity: $\frac{dk_x}{dt} = -\frac{\partial}{\partial x}\left(\underbrace{k_y\langle V_y\rangle}_{\text{mean}} + \underbrace{k_y\tilde{V}_y}_{\text{irregular, time varying}}\right)$

$\frac{d}{dt}\langle \delta k_x^2\rangle = D_k$

$$D_k = \sum_{q} q_x^2 k_y^2 |\tilde{V}_{y,q}|^2 \tau_{c\,k,q}$$

"random shearing"

"zonal field" (static)

Magnetic Field:

$$\frac{dk_x}{dz} = -\frac{k_y}{L_s} - \frac{\partial}{\partial x}\left(k_y\underbrace{\frac{\tilde{B}_y}{B_0}}_{\text{zonal field perturbation}}\right)$$

$\Rightarrow$

$$\frac{d}{dz}\langle \delta k_x^2\rangle = D_k'$$

$$D_k' = \sum_{q_x} q_x^2 k_y^2 \left|\frac{\tilde{B}_{y,q}}{B}\right|^2 l_{c\,k,q}$$

induce diffusive increase k_x.

Which brings us to:

→ Resistive Interchanges!

Recall:

- interchange ⟺ $k_{\parallel} = 0$
  (flute)

- stabilized by shear
  $k_{\parallel} = k_y x / L_s$ $\quad \left(\hat{E}_{\parallel} = 0 \right)$

- then introduce resistivity → decouple field and fluid;
  x scale and $k_{\parallel}$ linked.

$$\underbrace{\nabla_\perp^2 \hat{\phi} + \frac{V_A^2}{\gamma \eta} \nabla_{\parallel}^2 \hat{\phi}} + \frac{g}{|L_\rho| \gamma^2} k_y^2 \hat{\phi} = 0$$

$\nabla_{\parallel} = i k_{\parallel}(x)$ $\quad$ solve ala' QM H.O.

$$\gamma \sim \mathcal{O}(\eta^{1/3}) \to \mathcal{O}(1/S^{1/3})$$

$$1/S = a\eta / a^2 V_A$$

Spatial width ~ $O(a/s^{1/3})$.

$w \sim a/s^{1/3}$

$s \gg 1$. why? ⇒ shear / mode pinned to resonant surface (periodicity)

Now, consider twisting coordinates, clarity

$\boxed{\text{assume infinite length. (no b.c.)}}$

$$\chi = y - \frac{x}{L_s} z \qquad \varphi = z$$

y' $\qquad$ z'

$$\xi = x$$

x'

$\boxed{z', \varphi \text{ assume to extend } \pm\infty, \text{ along line. Does not satisfy boundary condition. Quasi-mode packet}}$

ballooning formalism reconciles twisted slice and periodicity!

↓

Key

and $\dfrac{\partial}{\partial y} \Rightarrow \dfrac{\partial}{\partial \chi}$

$$\frac{\partial}{\partial x} \Rightarrow \frac{\partial}{\partial \xi} - \frac{\varphi}{L_s}\frac{\partial}{\partial \chi}$$

$$\frac{\partial}{\partial z} \Rightarrow \frac{\partial}{\partial \varphi}$$

$$\boxed{\left[\left(\frac{\partial}{\partial \xi} - \frac{\varphi}{L_s}\right)^2 + \frac{\partial^2}{\partial x^2}\right]\hat{\phi} + \frac{V_A^2}{\gamma\eta}\frac{\partial^2\hat{\phi}}{\partial\varphi^2} - \frac{g}{L_{\rho}}\frac{\partial^2}{\partial x^2}\hat{\phi} = 0}$$

Lin Eqn. in twisted slicing coordinates:

line, radius, binormal

$$\hat{\phi} = \hat{\phi}(\varphi)\, e^{ik_x \xi}\, e^{ik_y \chi}$$

$$= e^{ik_x x}\, e^{+ik_y(y - \frac{x}{L_s}z)}\, \hat{\phi}(z)$$

'eigenfunction' describes length along field lines

opens door to "ballooning"

$$\frac{\partial^2\hat{\phi}}{\partial\varphi^2} - \gamma\eta\frac{k_y^2}{V_A^2}\left[\left(\frac{k_x}{k_y} - \frac{\varphi}{L_s}\right)^2 + 1\right]\hat{\phi}$$

$$+ \frac{k_y^2 g}{L_\rho}\frac{\gamma\eta}{V_A^2\gamma^2}\hat{\phi} = 0$$

$k_x << k_y$ (large scale)

differs from usual resistive interchange

$$\frac{\partial^2 \tilde{\phi}}{\partial \varphi^2} - \gamma \frac{\eta k_y^2}{V_A^2}\left[\frac{\varphi^2}{L_S^2} + 1\right]\tilde{\phi} + \frac{\eta k_y^2 \gamma_I^2}{\gamma V_A^2}\tilde{\phi} = 0$$

$$\tilde{\phi}(\varphi) = e^{-\alpha \varphi^2/2}$$

$\Rightarrow$

$\gamma \tau_A \sim S^{-1/3}$

$\gamma \sim \eta^{1/3}$

$\sqrt{\alpha} \sim \frac{1}{L_{eff}} \sim \eta^{1/3}$

$L_{eff} \sim \eta^{-1/3}$

effective length along field line

What is new?

$\Rightarrow$ $k_x a \sim 1$ i.e. no constraint on spatial scale,

$k_x << k_y$,

$\Rightarrow$ like slice

Comments

- $\vec{\phi} = f(x)\,\vec{\phi}(z)\, e^{i k_y (y - \frac{x}{L_s} z)}$

envelope in x, slow

eigenfunction $l_\parallel \sim \eta^{-1/3}$

twisting periodical variation

- here "modes" are finite length rolls, locally aligned with shear

ϕ const along $y - \frac{z x}{L_s} = \text{const.}$

- "modes" here are really $\boxed{\text{wave-packets}}$ of localized resistive interchange

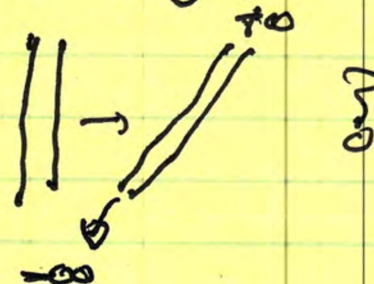
eigenmodes

"quasimodes"

"modes", "quasi-modes" → no b.c. in z, φ

modes → satisfy b.c. (periodicity) in ϕ.

- "quasimodes" have finite length in
  z → allow twist to align with shear
  → energetics!



- $\gamma \sim \eta^{1/3}$ but $k_x a \sim 1$
  → large transport i.e. $\frac{\gamma}{k_x^2} \sim \eta^{1/3}$

contrast: $\gamma \sim \eta^{1/3}$, $\Delta \sim \eta^{1/3}$

$$-\eta \frac{\partial p_0}{\partial r} = \Gamma_p$$

$\downarrow$

weak, $\frac{\gamma}{k_x^2} \sim \eta$.

- wave packets disperse, but relevant.

$$\boxed{\begin{array}{l}\text{Poloidal coupling due toroidicity} \\ \Rightarrow \text{wave packets become eigenmodes}\end{array}}$$

TBC

→ Toroidicity

- Toroidicity ⟺ $\boxed{\text{Ballooning}}$

- N.B.: Ballooning

  physics – tendency to localize fluctuations on bad curvature region → outside

  generic term for toroidal modes.

  Fluctuations / Modes where Toroidicity Matters.

- 

$$\begin{cases} k_\parallel = k_y x / L_s \\ \eta \to \text{reciprocal} \to \underbrace{k_y \hat{s} \eta}_{= k_r} \end{cases}$$

→ unfavorable curvature

but modes extend in η (i.e. along line → z) of twisted slice

$\boxed{\text{Key question: How reconcile toroidicity, shear, periodicity?}}$

– what happens with toroidicity?

→ Magnetic drifts enter!

→ propagator: $\omega - k_\parallel V_\parallel + \ldots \Rightarrow$
(particle)

$$\omega - \omega_{Di} - k_\parallel V_\parallel$$

↳ curvature

$$\omega_{Di} = \hat{\omega}_D \cos\theta + \tilde{\omega}_D \sin\theta \, \Delta \frac{\partial}{\partial r}$$

Curvature drift ⇔ couples poloidal harmonics!

ie if consider drift-acoustic modes:

→ no sound:

$$k_\perp^2 \rho_s^2 + 1 - \frac{\omega_*}{\omega} = 0$$

sound:

$$k_\perp^2 \rho_s^2 + 1 - \frac{\omega_*}{\omega} - \frac{k_\parallel^2 c_s^2}{\omega^2} = 0$$

drift – acoustic

Shear: $\rightarrow$ Eigenmode Eqn.

$$-\rho_s^2 \frac{\partial^2}{\partial x^2}\hat{\phi} + k_\theta^2 \rho_s^2 \hat{\phi} + \left(1 - \frac{\omega_*}{\omega}\right)\hat{\phi}$$

$$- \frac{k_\theta^2 x^2 c_s^2}{L_s^2 \omega^2}\hat{\phi} = 0$$

$\hat{\phi}_{m,n} = \hat{\phi}$

$\Rightarrow$ outgoing waves

$\Rightarrow$ absorbed at x_i

TBD

$\rightarrow$ Mode structure for drift wave (i.e. no instability $\rightarrow$ need non-adiabatic electrons!) in sheared system.

Toroidicity

$$-\rho_s^2 \frac{\partial^2}{\partial x^2}\hat{\phi}_m + k_\theta^2 \rho_s^2 \hat{\phi}_m + \left(1 - \frac{\omega_*}{\omega}\right)\hat{\phi}_m$$

$$- \frac{k_\theta^2 x^2 c_s^2}{L_s^2 \omega^2}\hat{\phi}_m + \underbrace{T}_{O\left(\frac{\omega_D}{\omega}\right)}\left(\hat{\phi}_{m+1} + \hat{\phi}_{m-1}\right) + \tilde{T}\left(\hat{\phi}_{m+1} - \hat{\phi}_{m-1}\right) = 0$$

Δ

$T, \bar{T}$ order ϵ_T.

so note that in toroidal system can write eigenmode equation as:

$$L_{\substack{m\\n}}\hat{\phi}_{\substack{m\\n}} + T_{\substack{m+1\\n}}(\phi_{m+1}) + \bar{T}'_{\substack{m-1\\n}}(\phi_{m-1}) = 0$$

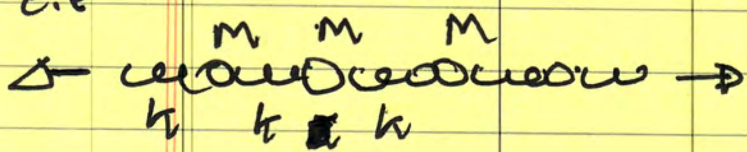
i.e.

- n remains good quantum #.
  - → Drifts don't break toroidal symmetry
- toroidal drifts tend to couple poloidal harmonics → Linearly.
- Eigenmode Eqn. is tri-diagonal matrix eqn.

$$\begin{bmatrix} T_{m-2} & L_{m-1} & \bar{T}_{m+2} & & \\ & \bar{T}_{m-1} & L_m & \bar{T}_{m+1} & \\ & & T_{m-2} & L_{m+1} & T_{m+2} \end{bmatrix} = 0$$

- Problem reduces to Bloch Eigenmode problem, à la solid state physics

i.e

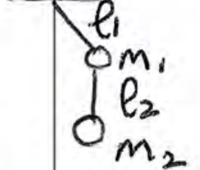


$\Rightarrow$ Toroidicity renders problem equivalent to linear chain, from elementary classical mechanics!

Linear Chains, etc.

→ Small Oscillations II — { Chains, Strings and the Transition Discrete → Continuous

— previously, considered few-degree-of-freedom systems, i.e.



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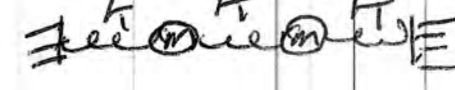
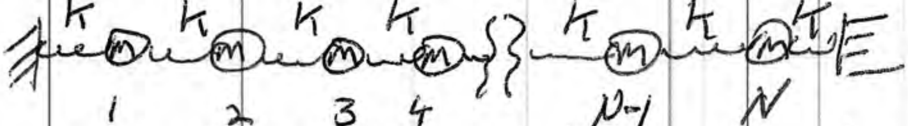


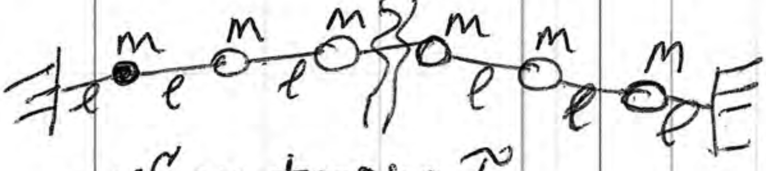
now, consider systems with $N \gg 1$ degrees of freedom, i.e. (separated by ℓ at equilibrium)

i) — linear chain (1D oscillators) (application → solid)



(identical components) → monatomic

ii) — massless string (loaded)



uniform tension T

uniform { mass m, separation ℓ

For i)

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m \dot{x}_i^2 - \left(\frac{1}{2} k (x_i - x_{i-1})^2 + \underbrace{\frac{1}{2} k (x_{i+1} - x_i)^2}_{\frac{1}{2}k(x_{i+1}-x_i)^2} \right) \right)$$

$$\begin{cases} x_0 \equiv 0 \\ x_{N+1} \equiv 0 \end{cases}$$

or simply

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m \dot{x}_i^2 - \frac{k}{2} (x_{i+1} - x_i)^2 \right)$$

Compressional Modes

For (ii),

$$L = \sum_{i=1} \left(\frac{1}{2} m \dot{y}_i^2 - \frac{\tau}{2\ell} (y_{i+1} - y_i)^2 \right)$$

transverse modes

~ identical systems.

∴ Hereafter, focus on i.)

motivations for i.) — monatomic chain is simplest example of elastic wave in solid

— step toward continuous system <u>i.e.</u> now discrete → masses separated by ℓ

Proceeding:

$$m \ddot{x}_i - k \left[(x_{i+1} - x_i) + (x_{i-1} - x_i) \right] = 0$$

$$\ddot{x}_i + \frac{k}{m} \left[2x_i - (x_{i+1} + x_{i-1}) \right] = 0$$

$$x_i = \hat{x}_i e^{-i\omega t}$$

$$\left(\frac{2k}{m}-\omega^2\right)\hat{x}_i - \frac{k}{m}\left(\hat{x}_{i-1}+\hat{x}_{i+1}\right) = 0$$

For eigenvalues, $\det \underline{\underline{A}} = 0$

$$\underline{\underline{A}} = \begin{vmatrix} \frac{2k}{m}-\omega^2 & -k/m & & & \bigcirc \\ -k/m & \frac{2k}{m}-\omega^2 & -k/m & & \\ & -k/m & \frac{2k}{m}-\omega^2 & -k/m & \\ \bigcirc & & -k/m & \frac{2k}{m}-\omega^2 & -k/m \end{vmatrix}$$

i.e. A tri-diagonal.

Now, taking masses separated by ℓ, take

$$\hat{x}_n \sim e^{i(n\ell)\underset{\text{wave-vector}}{\alpha}} \qquad \begin{cases} n \equiv \text{bead \#} \\ \alpha \equiv \text{wave \#} \\ \ell \equiv \underline{\text{spacing}} \end{cases}$$

$\Rightarrow$

$$\left(\frac{2k}{m}-\omega^2\right)e^{i\left[i\,\ell\alpha\right]} - \frac{k}{m}\left(e^{i\left[(i+1)\,\ell\alpha\right]} + e^{i\left[(i-1)\,\ell\alpha\right]}\right) = 0$$

careful i's.

$$\therefore \left(\frac{2k}{m}-\omega^2\right) - \frac{2k}{m}\cos\left[(\alpha\ell)\right] = 0$$

Note: says $\hat{x}_{n+m} = e^{\underset{\text{phase displ} \sim m\ell}{i m \ell\alpha}}\,\hat{x}_m$

Sol/ $\omega^2 = \frac{2k}{m}(2)\left[\frac{1-\cos(\alpha l)}{2}\right]$

$= \frac{4k}{m}\sin^2\left(\frac{\alpha l}{2}\right)$

$\Rightarrow \boxed{\omega^2 = \frac{4k}{m}\sin^2(\alpha l/2)}$

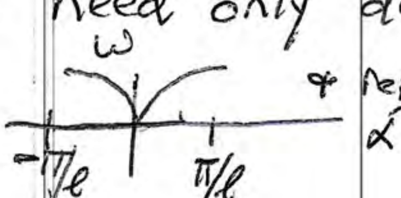
$\omega = 2\sqrt{k/m}\,|\sin \alpha l/2|$

Note:

① − $\omega = \omega_{max}\,|\sin(l\alpha/2)|$; $\omega_{max}^2 = 4k/m$

$\begin{cases} \omega(\alpha) = \omega(-\alpha) \\ \alpha' = \alpha + 2\pi/l \quad \text{leaves } \omega \text{ invariant} \end{cases}$

i.e. need only define α on $[-\pi/l, \pi/l]$ i.e. First Brillouin Zone, only needed



+ repeats

② − for $l\alpha/2 \ll 1$

i.e. wavelength $\alpha^{-1} \gg$ beadspacing l

→ continuum limit

then $\omega = \sqrt{k/m}\, \ell\, \alpha$

$= \alpha \left[\ell \sqrt{k/m}\right]$

akin to acoustic wave

$\omega = k\, C_s$

$$\begin{cases} k \leftrightarrow \alpha \\ C_s \leftrightarrow \ell\sqrt{k/m} \\ \dfrac{\gamma p}{\rho} \leftrightarrow \dfrac{\ell^2 k}{M} \end{cases}$$

$\ell^2 k$ → stored elastic energy (springiness)

M → inertia

③ – observe maximum frequency propagated is:

$\omega^2 = \omega_{max}^2 = 4k/m$ i.e. $\begin{cases} \omega^2 > \omega_{max}^2 & \text{not propagated} \\ \omega^2 < \omega_{max}^2 & \text{propagated} \end{cases}$

Chain acts as low-pass filter
Higher frequencies evanescent!

④ – for propagation structure;

$\omega = 2\sqrt{k/m}\, \left[\sin(\alpha \ell/2)\right]$

$V_{gr} = d\omega/d\alpha = \ell\sqrt{k/m}\, \cos(\alpha \ell/2)$

i.e. $V_{gr} = \ell\sqrt{k/m} \sim C_{s\,eff}$ for $\alpha \ell \lll 1$
(a la' sound)

but $\lim_{\alpha \to \pi/\ell} v_{gr} = \ell\sqrt{k/m}\,\cos(\pi/2) \to 0$

i.e. modes at edge of Brillouin zone non-propagating

modes in middle of zone propagate at acoustic speed

Can also observe that:

$$x_{i+1} + x_{i-1} - 2x_i = e^{i[\alpha\ell]i}\left(e^{i\alpha\ell} + e^{-i\alpha\ell} - 2\right)$$

$$= 2e^{i[i\alpha\ell]}\left(\cos\alpha\ell - 1\right)$$

so $\cos\alpha\ell / 1 \sim$ ratio of $(x_{i+1} + x_{i-1})/2x_i$

$\sim$ mean phase ratio

so $\alpha\ell \ll 1$ (in zone) $\Rightarrow$ neighbors on chain vibrate <u>in</u> phase $\to$ propagation
<u>cos</u> $= 1$

$\alpha\ell \sim \pi$ (zone boundary) $\Rightarrow$ neighbors on chain vibrate <u>out</u> <u>of</u> phase $\to$ no propagation

$\cos = -1$

What if?:

→ Boundary Conditions

Can distinguish 2 cases $\begin{cases} \text{periodic B.C.'s} \\ \text{fixed end B.C.'s.} \end{cases}$

1) Periodic B.C.'s

Now, $x_i = Ae^{i[i\ l\alpha]}$

notational clarity ⇒ $x_n = Ae^{i[n l\alpha]}$

$1 < n < N.$

For periodic B.C.'s,

$x_n = x_{n+N} \Rightarrow e^{iNl\alpha} = 1$

↳ mode indx.

$\therefore \quad Nl\alpha = 2\pi p$

$$\Rightarrow \boxed{\alpha = \frac{2\pi p}{Nl}} \qquad p = \begin{cases} 0,\ \pm 1,\ \ldots \pm \frac{1}{2}(N-1) & N \text{ odd} \\ 0,\ \pm 1,\ \ldots\ \frac{1}{2}N & N \text{ even} \end{cases}$$

Note: guarantees N normal modes.

2, Fixed end B.C.'s: $\left.\begin{array}{l} x_0 = 0 \\ x_{N+1} = 0 \end{array}\right\}$ guarantees ends fixed

$$\Rightarrow \quad x_0 = x_{N+1} = 0$$

$$x_n = Ae^{in\alpha l} + Be^{-in\alpha l}$$

$$= A\sin(nl\alpha) + B\cos(nl\alpha)$$

$$B = 0 \rightarrow n = 0 \checkmark$$

$$(N+1)\alpha l = p\pi \quad ; \quad p = 1, \ldots N$$

mode index

$$\Rightarrow \quad \boxed{\alpha_p = p\pi / l(N+1)}$$

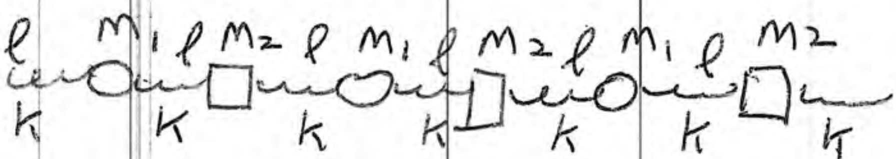
{acts to quantize k

$$\therefore \quad x_n(t) = A_n \sin\left(\frac{n \cancel{l} p\pi}{\cancel{l}(N+1)}\right) e^{-i\omega_p t}$$

$$\text{where} \quad \omega_p^2 = \frac{4k}{m} \sin^2\left(\frac{p\pi \cancel{l}}{2\cancel{l}(N+1)}\right)$$

Diatomic Chain

→ consider slightly richer toy model, namely the diatomic chain



{un-equal masses!

then, no loss of generality to associate

$m_1 \rightarrow x_{2n}$ (evens)
$m_2 \rightarrow x_{2n+1}$ (odds) } positions

∴ can immediately write dynamical equations

$$m_1 \ddot{x}_{2n} = -k\,(2x_{2n} - x_{2n-1} - x_{2n+1})$$

$$m_2 \ddot{x}_{2n+1} = -k\,(2x_{2n+1} - x_{2n} - x_{2n+2})$$

solution of form:

$$x_{2n} = A e^{i n \ell \alpha} e^{-i\omega t} \qquad \text{(evens)}$$

$$x_{2n+1} = B e^{i(2n+1)\ell\alpha} e^{-i\omega t} \qquad \text{(odds)}$$

(consider one mass → d, co)

$$-m_1\omega^2 A = -k\left(2A - \left(e^{il\alpha} + e^{-il\alpha}\right)B\right)$$

$$-m_2\omega^2 B = -k\left(2B - \left(e^{il\alpha} + e^{-il\alpha}\right)A\right)$$

$\Rightarrow$

$$(-m_1\omega^2 + 2k)A - k(2\cos l\alpha)\,B = 0$$

$$(-2k\cos l\alpha)A + (-m_2\omega^2 + 2k)\,B = 0$$

$\therefore$

$$\boxed{(\omega^2 - 2k/m_1)(\omega^2 - 2k/m_2) - \frac{4k^2}{m_1 m_2}\cos^2 l\alpha = 0}$$

$\Rightarrow$ dispersion relation:

$$\omega^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm k\left\{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 - \frac{4\sin^2(\alpha l)}{m_1 m_2}\right\}^{1/2}$$

$1/\mu \equiv 1/m_1 + 1/m_2$ $\Rightarrow$ reduced mass, as usual.

$$\omega^2 = k/\mu \pm k/\mu\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}$$

$\therefore$ dispersion relation:

$$\boxed{\omega^2 = \frac{k}{\mu}\left\{1 \pm 1\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}\right\}}$$

can immediately observe:

→ system supports 2 modes

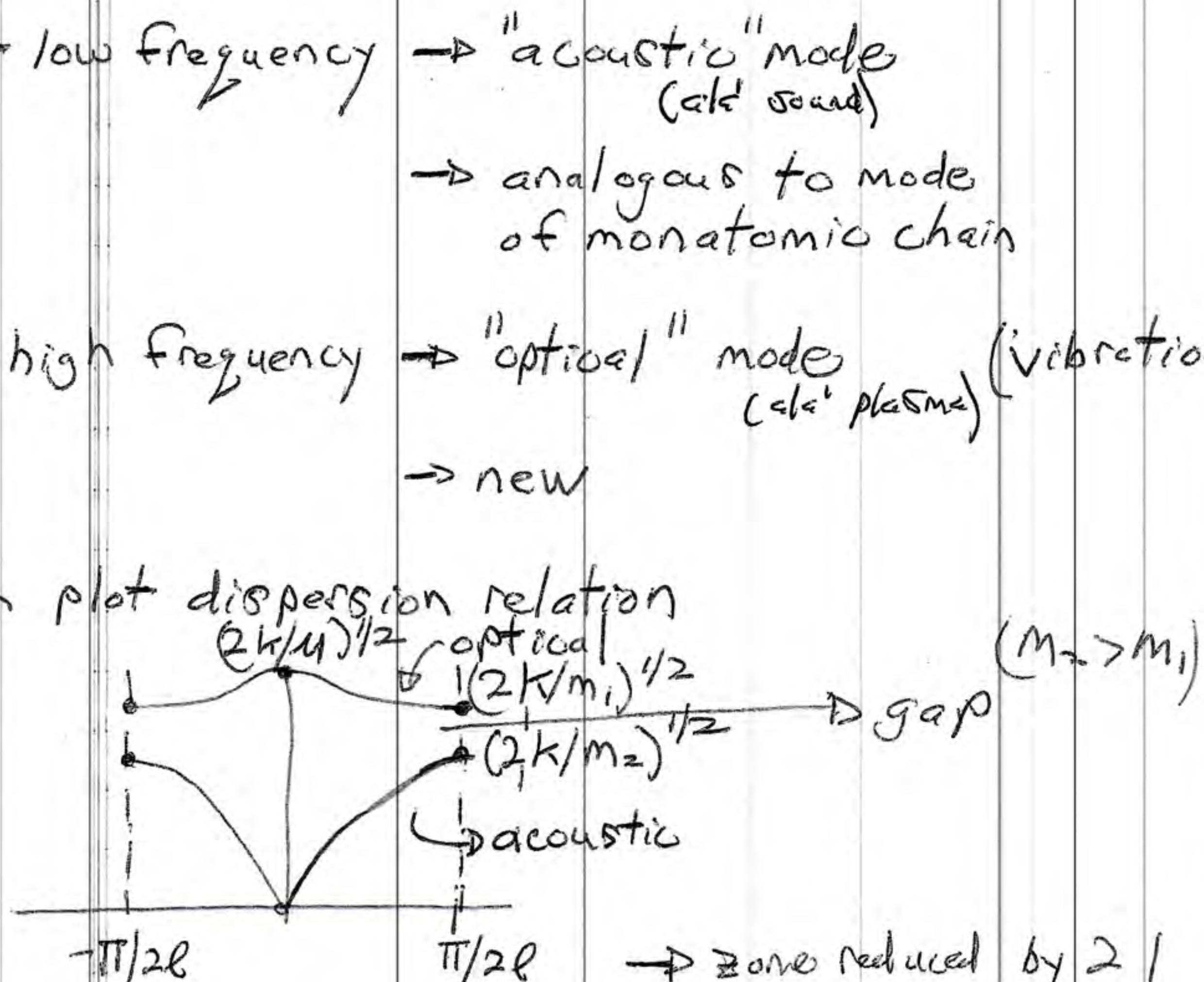
- low frequency → "acoustic" mode (ala' sound)

→ analogous to mode of monatomic chain

- high frequency → "optical" mode (ala' plasma) (vibration)

→ new

Can plot dispersion relation



→ zone reduced by 2! (scale)

Note: - acoustic mode $\omega \sim \alpha \left(\frac{k\ell^2}{m_2 + m_1} \right)^{1/2}$

$\sim k C_{s_{eff}}$

as $k\ell \to 0$ ⇒ mass neighbors vibrate <u>in phase</u>

$x_n = x_{n+1}$

solid → phonon ($\omega = kC_s$)

optical mode $\omega \sim (2k/\mu)^{1/2}$

as $kl \to 0$; $m_1 x_n + m_2 x_{n+1} = 0$
i.e. neighboring masses vibrate
out of phase, weighted by
masses

solid → analogous collective mode is EM wave
$\omega^2 = \omega_{pe}^2 + c^2 k^2$ or plasmon
$\omega^2 = \omega_{pe}^2 + k^2 v_{Te}^2$

i.e. $k \to 0$, frequency constant!

→ Note gap → no propagation for
$$(2k/m_2)^{1/2} < \omega < (2k/m_1)^{1/2}$$
→ Consequence of fact
phonon ↔ inertia of heavy mass
optical ↔ inertia of light mass
(in ω_p^2)

→ Transition to Continuum

To recover continuum $\left\{ \text{i.e.} \begin{array}{l} \text{elastic medium} \\ \text{massive string} \end{array} \right.$

take $\begin{array}{l} N \to \infty \\ \ell \to 0 \end{array}$ with constant $L = (N+1)\ell$

$$\left\{ \begin{array}{l} \frac{m}{\ell} = \mu = \text{const.} \\ k\ell = K = \text{const.} \end{array} \right.$$

Note: "$N \to \infty$" means $N > p$, for all modes p.

Then;

$$\omega_p^2 = \frac{4k}{m} \sin^2\left(\frac{p\pi}{2(N+1)}\right)$$

$$\cong \frac{4k}{m}\left(\frac{p\pi}{2(N+1)}\right)^2$$

$$= \frac{(p\pi)^2}{((N+1)\ell)^2} \frac{k\ell^2}{m}$$

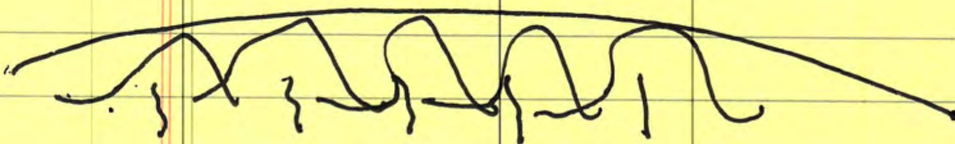
$$= \left(\frac{p\pi}{L}\right)^2 \left(\frac{K}{\mu}\right) = \left(\frac{p\pi}{L}\right)^2 c_s^2$$

$$c_s^2 = k\ell^2/m = (k\ell)\,\ell/m = K/\mu$$

$$\to \quad \omega^2 = k^2 c_s^2 \quad ; \quad \begin{array}{l} c_s^2 = K/\mu \\ k = p\pi/L \end{array}$$

→ How to crank?

- notes



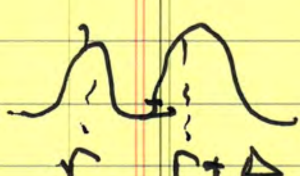
→ Bloch eigenfunction = superposition of coupled (linearly) poloidal harmonics. Like chain.

→ Like chain → there have definite phase relation between them.

→ like chain, lowest (most unstable) mode, ⇒ harmonics <u>in</u> phase.

→ <u>spacing</u>: $q = m/n$

$$q(r+\Delta r) = (m+1)/n$$



$$\boxed{\Delta r = 1/nq'}$$

$$k_\theta \Delta r = \frac{k_\theta \hat{s}}{R_E}\,\frac{1}{k_\theta \hat{s}}$$
$$= 1/R_E .$$

$$\Delta r = 1/n q'$$
$$= 1\Big/\frac{nq}{r}\,\frac{q' r}{q} = 1/k_\theta \hat{s}$$

→ fundamental scale induced by toroidal coupling

→ physically → scale of spacing of poloidal harmonics at fixed n.

– Translation invariance:

$$L_{m,n}\,\hat{\phi}_{m,n}(x) + T_{m+1,n}\,\hat{\phi}_{m+1,n}(x') + T_{m-1,n}\,\hat{\phi}_{m-1,n}(x'')$$
$$+ \dots = 0$$

<u>i.e.</u>

$$\hat{\phi}_{m+1} = f\left(x' + r_{m+1,n}\right)$$

distance from surfaces displaced by Δ.

$$\hat{\phi}_{m-1} = f\left(x'' + r_{m-1,n}\right)$$

Then assume translation invariance:

$$\hat{\phi}_{\underset{n}{m+1}} = \hat{\phi}_{\underset{n}{m}}(x+\Delta)$$

$\longrightarrow$ shift by Δ

$$\hat{\phi}_{\underset{n}{m-1}} = \hat{\phi}_{\underset{n}{m}}(x-\Delta)$$

So eigenmode equation becomes:

couplings

$$L_{\underset{n}{m}}\hat{\phi}_{\underset{n}{m}}(x) + T_{\underset{n}{m+1}}\hat{\phi}_{\underset{n}{m}}(x+\Delta) + T_{\underset{n}{m-1}}\hat{\phi}_{\underset{n}{m}}(x-\Delta) + \ldots = 0$$

Now: - tridiagonal matrix equation converted to differential difference equation

- simply Fourier transform in x

N.B. F.T. $(x) \rightarrow$ distance along field line η.

- solve eigenmode equation on $\parallel$

$\Rightarrow$ determines how mode varies along field line $\Leftrightarrow$ analogue of quasimode.

- like Bloch envelope - determine radial envelope from envelope B.C.

N.B:

- scale comparison: Δ^{-1} vs ϕ'/ϕ

$\Delta < (\phi'/\phi)^{-1}$ $\Rightarrow$ "strong" ballooning

$\Rightarrow$ expand in Δ

$\Leftrightarrow$ drive enough so mode broadens.

$\Delta\phi'/\phi > 1$ $\Rightarrow$ "weak" ballooning
can't expand

$\Rightarrow$ mode extends in $\perp$.

→ Fancy approach: "Ballooning Mode Representation"

Connor, Hastie, Taylor '78 → {Gala event of late 70's in MFE}

$$\varphi_n(\theta, x) = \sum_m e^{-im\theta} \int_{-\infty}^{+\infty} e^{im\eta} \hat{\phi}(\eta, x)\, d\eta$$

→ sum of poloidal harmonics

→ η is along field line (F.T. of $k_{\parallel}$, and thus x)

→ $\hat{\phi}$ not periodic
φ is periodic } periodicity issue resolved.

→ can take $\hat{\phi} \sim \hat{\phi}\, e^{-inq\eta}$

eikonal for fast variation, in η.

→ Comes to same thing as Bloch approach.

(Why the noise?. [Robts + Taylor '65 → Physics])

→ What happens for case of drift wave?

Shear:

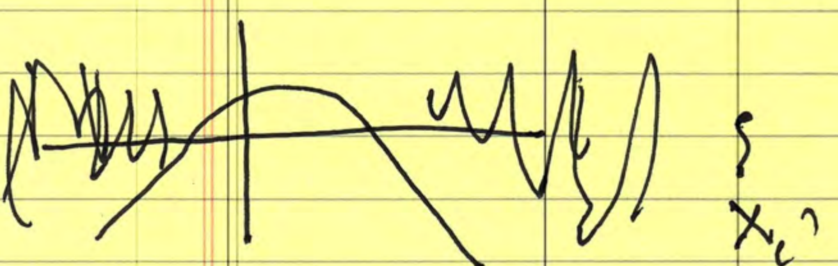
$$-\rho_s^2 \frac{d^2}{dx^2}\tilde{\phi} + k_\theta^2 \rho_s^2 \tilde{\phi} + \left(1 - \frac{\omega_*}{\omega}\right)\tilde{\phi} - \frac{k_\theta^2 x^2 c_s^2}{L_s^2 \omega^2}\tilde{\phi} = 0$$

note:

$$\frac{d^2}{dx^2}\tilde{\phi} + \frac{k_\theta^2 x^2 c_s^2}{L_s^2 \omega^2}\tilde{\phi} + (\ ) = 0$$

anti-well! ⇒ outgoing wave.

$\tilde{\phi} \approx e^{-i\mu x^2/2}$ ⇒ energy propagated to absorption at x_i



$\omega = k_\parallel v_{thi}$

$x_i = L_s \omega / k_\theta v_{thi}$

⇒ here appears as radiative damping!

⇒ magnetic shear damping.

$$\frac{\delta\omega}{\omega} \sim -i\,\theta_s^3 \mu$$

→ Physics : ion Landau damping at x_i

With Toroidicity

Mode potential. shear

$$\underbrace{\frac{\partial^2 \hat{\phi}}{\partial \eta^2} + (\,)\,\eta^2 \hat{\phi}}_{\text{FT of Gaussian is Gaussian!}} + \frac{\omega_D}{\omega}\left(\cos\eta + \hat{s}\,\eta \sin\eta\right)\hat{\phi} + (\,)\,\hat{\phi} = 0$$

curvature

curvature pieces → ripples,
can localize modes in
ripples

⇒ beats shear damping !

→ "Toroidicity Induced Mode."
chen, cheng '80

Allows ⊖ marginal drift wave ⇒ easily avoids magnetic shear damping

→ What of $\hat{s} \to 0$? $\boxed{\text{Flat } q}$ → JET (TFTR, DIII-D, EAST, KSTAR)

- fewer surfaces, i.e. $\Delta \uparrow$
- ballooning structure ↓
- $\hat{s}\, \eta \sin \eta \to 0$

⇒ ballooning structure collapses

⇒ curvature (drive effects) weakened

⇒ confinement improvement.

⇒ E_r shear ? ...