PHYSICS 211B : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #3

(1) Define the operator

$$\Pi_N = \frac{1}{N!} \int\limits_{\mathbb{R}^{dN}} d^d x_1 \cdots d^d x_N \, | \, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \, \rangle \langle \, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \, | \quad ,$$

where

$$| \, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \, \rangle = \psi^\dagger(\boldsymbol{x}_1) \cdots \psi^\dagger(\boldsymbol{x}_N) \, | \, 0 \, \rangle \quad ,$$

where $\big[\psi(\boldsymbol{x}) \, , \, \psi^\dagger(\boldsymbol{x}')\big]_\mp = \delta(\boldsymbol{x} - \boldsymbol{x}')$ for bosons $(-)$ and fermions $(+)$. Here each $\boldsymbol{x}_j \in \mathbb{R}^d$.

(a) Show that Π_N is a projector onto the totally symmetric and totally antisymmetric parts of the N-body Hilbert space for bosons and fermions, respectively.

(b) Show that one can also write

$$\Pi_N \equiv \int\limits_{\Delta_N} d^d x_1 \cdots d^d x_N \, | \, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \, \rangle \langle \, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \, | \quad ,$$

where Δ_N is defined to be the subset of $\mathbb{R}^{dN}$ for which

$$\Delta_N = \Big\{ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \, | \, x_1^{(1)} < x_2^{(1)} < \cdots < x_N^{(1)} \Big\} \quad .$$

(2) Consider a one-dimensional electron gas with spin-independent interactions

$$u(x - x') = \frac{u_0}{\pi} \, \frac{\lambda}{(x - x')^2 + \lambda^2} \quad .$$

Find the Hartree-Fock energies $\varepsilon(k)$.

(3) For *spinless* electrons interacting via a potential $u(\boldsymbol{x})$, find the Hartree-Fock energies $\varepsilon(\boldsymbol{k})$. Show that when $\hat{u}(\boldsymbol{k}) = \text{const.}$ that there is no interaction contribution to $\varepsilon(\boldsymbol{k})$. Interpret this physically.

(4) Consider a polarized electron gas (three dimensions, Coulomb interactions) in which N_σ denotes the number of electrons with spin polarization σ.

(a) Begin with the Hamiltonian

$$\begin{aligned} \hat{H} = -\frac{\hbar^2}{2m} \sum_\sigma \int d^3x \, c^\dagger(\boldsymbol{x}) \, \nabla^2 \, c(\boldsymbol{x}) + \frac{1}{2} \sum_{\sigma,\sigma'} \int d^3x \int d^3x' \; c^\dagger_\sigma(\boldsymbol{x}) \, c^\dagger_{\sigma'}(\boldsymbol{x}') \, u(\boldsymbol{x} - \boldsymbol{x}') \, c_{\sigma'}(\boldsymbol{x}') \, c_\sigma(\boldsymbol{x}) \\ - \sum_\sigma \int d^3x \, c^\dagger_\sigma(\boldsymbol{x}) \, c_\sigma(\boldsymbol{x}) \int d^3x' \; u(\boldsymbol{x} - \boldsymbol{x}') \, n_0 + \frac{1}{2} \int d^3x \int d^3x' \; n_0 \, u(\boldsymbol{x} - \boldsymbol{x}') \, n_0 \end{aligned}$$

where $n_0 = N_0/V$ is the background number density and where $u(\boldsymbol{r}) = (e^2/r)\, e^{-Qr}$. is the Yukawa potential. *At the appropriate time, you may take the $Q \to 0$ limit in order to recover the jellium system.* Using the relation

$$c_\sigma(\boldsymbol{x}) = V^{-1/2} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, c_{\boldsymbol{k},\sigma}$$

show that one may write

$$\hat{H} = \sum_{\boldsymbol{k},\sigma} \varepsilon(\boldsymbol{k})\, c^\dagger_{\boldsymbol{k},\sigma}\, c_{\boldsymbol{k},\sigma} - n_0\, \hat{u}(0) \sum_{\boldsymbol{k},\sigma} c^\dagger_{\boldsymbol{k},\sigma}\, c_{\boldsymbol{k},\sigma} + \frac{1}{2V} \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \sum_{\sigma,\sigma'} \hat{u}(\boldsymbol{q})\, c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}\, c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}\, c_{\boldsymbol{p},\sigma'}\, c_{\boldsymbol{k},\sigma} + E_{\text{bg}}$$

with

$$\varepsilon(\boldsymbol{k}) = \frac{\hbar^2 \boldsymbol{k}^2}{2m} \quad , \quad \hat{u}(\boldsymbol{q}) = \frac{4\pi e^2}{\boldsymbol{q}^2 + Q^2} \quad , \quad E_{\text{bg}} = \frac{2\pi e^2}{Q^2}\, \frac{N_0^2}{V} \quad .$$

You may assume periodic boundary conditions in a $L \times L \times L$ box of volume $V = L^3$ in the limit $L \to \infty$. The allowed $\boldsymbol{k}$ values are then quantized according to $\boldsymbol{k} = \frac{2\pi}{L}\,(n_x, n_y, n_z)$ where $n_{x,y,z} \in \mathbb{Z}$.

(b) Find the ground state energy to first order in the interaction potential as a function of $N = N_\uparrow + N_\downarrow$ and the magnetization $M = N_\uparrow - N_\downarrow$. You should assume a wavefunction

$$|\,\Psi\,\rangle = \prod_{|\boldsymbol{k}|<k_{\mathsf{F}\uparrow}} c^\dagger_{\boldsymbol{k},\uparrow} \prod_{|\boldsymbol{k}'|<k_{\mathsf{F}\downarrow}} c^\dagger_{\boldsymbol{k}',\downarrow}\,|\,0\,\rangle \quad .$$

where $n_\sigma = k^3_{\mathsf{F},\sigma}/6\pi^2 = N_\sigma/V$ is the number density of electrons of spin polarization σ. Along the way, show that

$$\langle\,\Psi\,|\, c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}\, c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}\, c_{\boldsymbol{p},\sigma'}\, c_{\boldsymbol{k},\sigma}\,|\,\Psi\,\rangle = n_{\boldsymbol{k},\sigma}\, n_{\boldsymbol{p},\sigma'}\, \delta_{\boldsymbol{q},0} - n_{\boldsymbol{p},\sigma}\, n_{\boldsymbol{k},\sigma}\, \delta_{\boldsymbol{q},\boldsymbol{p}-\boldsymbol{k}}\, \delta_{\sigma,\sigma'} \quad ,$$

where $n_{\boldsymbol{k},\sigma} = \langle\,\Psi\,|\, c^\dagger_{\boldsymbol{k},\sigma}\, c_{\boldsymbol{k},\sigma}\,|\,\Psi\,\rangle$. Express your result for the energy as $E(n, \zeta, V)$, where $\zeta \equiv (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow)$ is the dimensionless magnetization and $n = N/V = n_\uparrow + n_\downarrow$.

(c) Prove, to this order in the interaction, that the ferromagnetic state ($M = N$) has a lower energy than the unmagnetized state ($M = 0$) provided r_{s} exceeds a critical value $r_{\mathsf{s},1}$. Find that critical value $r_{\mathsf{s},1}$.

(d) Define $\varepsilon(\zeta) = E/N$ with $\zeta = M/N$. Show that $\varepsilon''(0) < 0$ when r_{s} exceeds a critical value $r_{\mathsf{s},2}$. Find $r_{\mathsf{s},2}$. You should find $r_{\mathsf{s},1} < r_{\mathsf{s},2}$. What happens for $r_{\mathsf{s}} \in [r_{\mathsf{s},1}, r_{\mathsf{s},2}]$?