Lecture 10 (Feb. 4) : Fermi liquids at low temperatures

The first law says

$$
\begin{aligned}
T\,\delta S &= \sum_{k,\sigma} (\tilde{\varepsilon}_{k\sigma} - \mu)\, \delta n_{k\sigma} \\
&= \sum_{k,\sigma} (\tilde{\varepsilon}_{k\sigma} - \mu) \left\{ \frac{\partial n_{k\sigma}}{\partial \tilde{\varepsilon}_{k\sigma}}\, \delta\tilde{\varepsilon}_{k\sigma} + \frac{\partial n_{k\sigma}}{\partial \mu}\, \delta\mu + \frac{\partial n_{k\sigma}}{\partial T}\, \delta T \right\} \\
&= \sum_{k,\sigma} (\tilde{\varepsilon}_{k\sigma} - \mu) \left(\frac{\partial n_{k\sigma}}{\partial \tilde{\varepsilon}_{k\sigma}} \right) \left\{ (\delta\tilde{\varepsilon}_{k\sigma} - \delta\mu) - \left(\frac{\tilde{\varepsilon}_{k\sigma} - \mu}{T} \right) \delta T \right\}
\end{aligned}
$$

The first term (in green) turns out to contribute a piece of order $T^3 \ln T$, which we will ignore here. Thus

$$
\delta S = -\sum_{k,\sigma} \left(\frac{\partial n_{k\sigma}}{\partial \tilde{\varepsilon}_{k\sigma}} \right) (\tilde{\varepsilon}_{k\sigma} - \mu)^2 \, \frac{\delta T}{T^2} = \frac{\pi^2}{3}\, V g(\varepsilon_F)\, k_B^2\, \delta T
$$

(See Eqn. 10.44.) We conclude that

$$
c_V = \frac{T}{V} \left(\frac{\partial S}{\partial T} \right)_{V,N} = \frac{\pi^2}{3}\, g(\varepsilon_F)\, k_B^2\, T
$$

What differs from the IFG is that $g(\varepsilon_F) = \dfrac{m^* k_F}{\pi^2 \hbar^2}$, so we have freedom to fit m^* from the c_V data, assuming it is linear in T, and given $k_F = (3\pi^2 n)^{1/3}$. In other words, if $c_V^0(T)$ is the specific heat for an ideal Fermi gas of individual mass m, then

$$
\frac{c_V(T)}{c_V^0(T)} = \frac{m^*}{m}
$$

From $\delta F|_{V,N} = -S\,\delta T$, we integrate to obtain

$$F(T,V,N) = E_0(V,N) + \frac{\pi^2}{6} V g(\varepsilon_F)(k_B T)^2 + \ldots$$

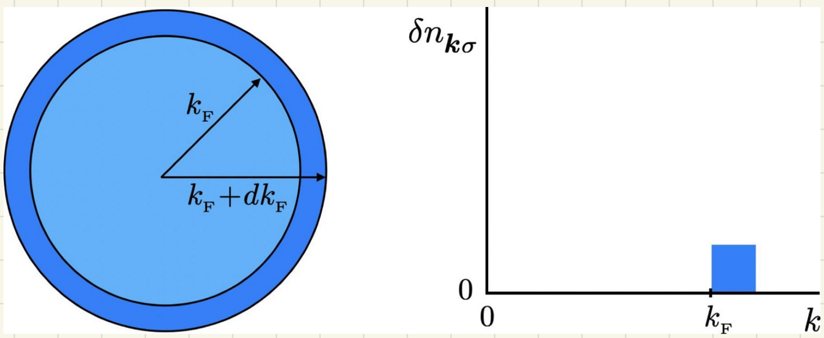
from which we obtain

$$\mu(T,n) = -\frac{\partial F}{\partial N}\bigg|_{T,V} = \mu(0,n) - \frac{\pi^2}{4}\left(\frac{1}{3} + \frac{\partial \ln m^*}{\partial \ln n}\right)\frac{k_B T^2}{T_F} + \ldots$$

with $k_B T_F = \hbar^2 k_F^2 / 2m^*$.

- **Compressibility and sound velocity**

Consider a swollen Fermi surface of radius $k_F + dk_F$:



The change in μ is then $d\mu = \tilde{\varepsilon}_{k_F+dk_F} - \tilde{\varepsilon}_{k_F} = d\tilde{\varepsilon}_F$, so

$$\begin{aligned} d\mu &= d\varepsilon_{k_F} + \frac{1}{V}\sum_{k',\sigma'} f_{k_F\sigma, k'\sigma'}\,\delta n_{k'\sigma'} \\ &= \hbar v_F \left\{1 + \sum_{\sigma'}\int \frac{d^3k'}{(2\pi)^3} f_{k_F\sigma,k'\sigma'}\,\delta(\varepsilon_{k'} - \mu)\right\} dk_F \\ &= \hbar v_F \left\{1 + 2\int \frac{d\Omega}{4\pi} f^s(\vartheta)\int \frac{d^3k'}{(2\pi)^3}\delta(\varepsilon_{k'} - \mu)\right\} \\ &= \hbar v_F (1 + F_0^s)\, dk_F \end{aligned}$$

Thus,

$$\kappa = n^{-2}\frac{\partial n}{\partial \mu} = \frac{n^{-2}g(\varepsilon_F)}{1+F_0^s} \Rightarrow \frac{\kappa}{\kappa^0} = \frac{m^*}{m}\cdot\frac{1}{1+F_0^s}$$

where $\kappa^0 = n^{-2}g_0(\varepsilon_F)$. Thus, κ/κ^0 depends on both m^*/m as well as on the Landau parameter F_0^s. In the inviscid weak flow limit of the Navier-Stokes eqns, $\partial_t(\rho\vec{u}) = -\vec{\nabla}p$ where $\rho = mn$ is the mass density. Here m is the **bare** mass (e.g., m_3) and p the pressure. From local thermodynamics, with T fixed,

$$\vec{\nabla}p = \frac{\partial p}{\partial \rho}\vec{\nabla}\rho = \frac{\vec{\nabla}\rho}{\kappa\rho}$$

hence

$\vec{\nabla}\cdot\vec{j} + \partial_t\rho = 0$ with $\vec{j} = \rho\vec{u}$ (continuity eqn.)

$$\vec{\nabla}\cdot\left(\frac{1}{\kappa\rho}\vec{\nabla}\rho\right) = -\partial_t\,\vec{\nabla}\cdot(\rho\vec{u}) = +\partial_t^2\rho$$

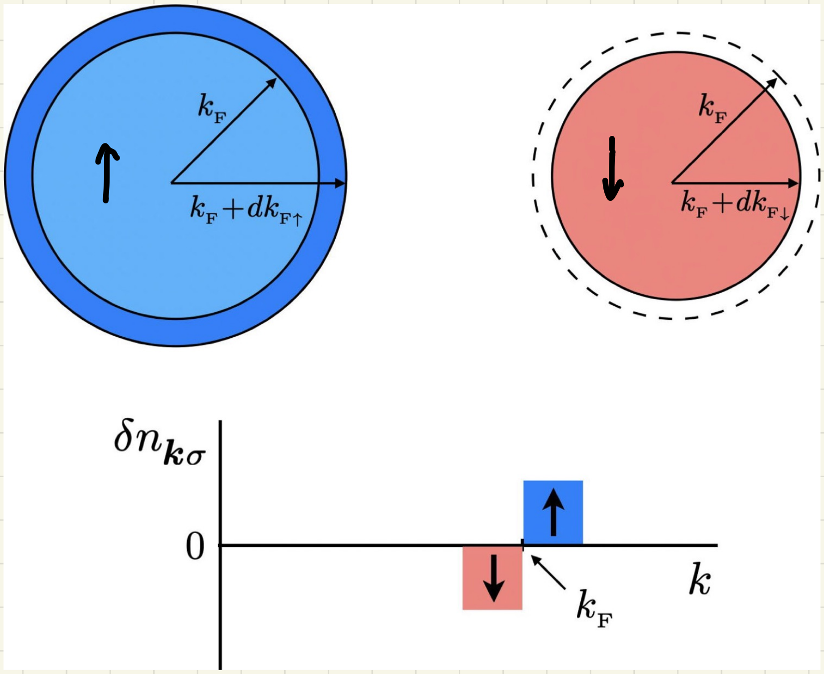
$$\Rightarrow \nabla^2\rho = \frac{1}{s^2}\frac{\partial^2\rho}{\partial t^2}$$

with $s = (\kappa\bar{\rho})^{-1/2}$ and $\bar{\rho}$ the mean density.

- **Uniform magnetic susceptibility**

In the presence of a uniform magnetic field B, there is an additional Zeeman term in the Hamiltonian: $H_Z = -\mu_0 B\sum_{k,\sigma}\sigma n_{k\sigma}$. This causes the ↑ Fermi surface to expand and the ↓ surface to contract. We write $dk_{F\uparrow} = -dk_{F\downarrow} \equiv dk_F$ and

$$\delta n_{k\sigma} = \sigma\,\delta(k-k_F)\,dk_F$$



To lowest nontrivial order in B, the chemical potential is not changed, and

$$d\tilde{\mathcal{E}}_{k_F\sigma} = -\sigma\mu_0\, dB + d\mathcal{E}_{k_F\sigma} + \frac{1}{V}\sum_{k',\sigma'} f_{k_F\sigma,\, k'\sigma'}\,\delta n_{k'\sigma'}$$

$$= -\sigma\mu_0\, dB + \hbar v_F\left\{\sigma + \sum_{\sigma'}\int\frac{d^3k'}{(2\pi)^3} f_{k_F\sigma,\, k'\sigma'}\,\sigma'\delta n_{k'\sigma'}\right\}dk_F$$

$$= -\sigma\mu_0\, dB + \sigma\hbar v_F\left\{1 + g(\varepsilon_F)\int\frac{d\Omega}{4\pi} f^a(\vartheta)\right\}dk_F$$

$$= -\sigma\mu_0\, dB + \sigma\hbar v_F(1+F_0^a)\,dk_F$$

Hence

$$\frac{\partial k_F}{\partial B} = \frac{\mu_0}{\hbar v_F(1+F_0^a)}$$

and

$$\chi = \frac{1}{V}\frac{\partial M}{\partial B}\bigg|_{B=0} = \mu_0\left(\frac{\partial n_\uparrow}{\partial k_{F\uparrow}} + \frac{\partial n_\downarrow}{\partial k_{F\downarrow}}\right)\left(\frac{\partial k_F}{\partial B}\right)_{B=0} = \frac{\mu_0^2\, g(\varepsilon_F)}{1+F_0^a}$$

We conclude

$$\frac{\chi}{\chi^0} = \frac{m^*/m}{1+F_0^a}$$

where $\chi^0 = \mu_0^2\, g_0(\varepsilon_F)$. Thus another Landau parameter enters, this time F_0^a. Landau parameters: F_L^ν ; $\nu \in \{s, a\}$, $L \in \{0, 1, \ldots\}$

- **Galilean invariance**

In a frame K' moving at velocity $\vec{u}$ relative to a fixed inertial frame K. The Hamiltonian in K' is

$$H' = \sum_i \frac{(\vec{P}_i - m\vec{u})^2}{2m} + \hat{H}_1 = \hat{H} - \vec{u}\cdot\vec{P} + \tfrac{1}{2} M\vec{u}^{\,2}$$

total momentum

total mass

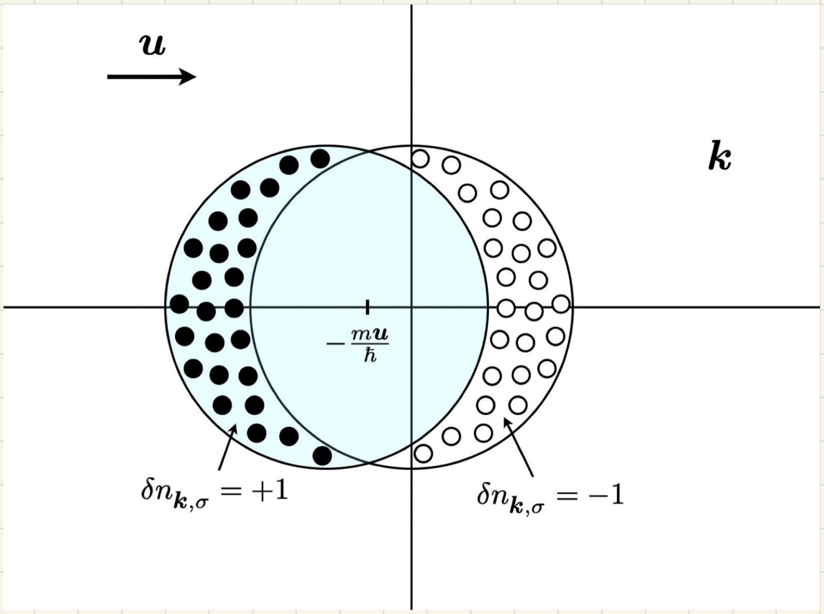
Suppose we add a particle of momentum $\vec{p} = \hbar\vec{k}$ and spin polarization σ in the lab frame (K) at $T=0$. Its energy is $\varepsilon_{\vec{k}\sigma}$. In the primed frame, its energy

$$\tilde{\varepsilon}'_{\vec{k}-\hbar^{-1}m\vec{u},\sigma} = \varepsilon_{\vec{k}\sigma} - \hbar\vec{k}\cdot\vec{u} + \tfrac{1}{2} m\vec{u}^{\,2}$$

and thus

$$\tilde{\varepsilon}'_{\vec{k}\sigma} = \varepsilon_{\vec{k}+\hbar^{-1}m\vec{u},\sigma} - \hbar\vec{k}\cdot\vec{u} - \tfrac{1}{2} m\vec{u}^{\,2}$$

Now let's calculate $\tilde{\varepsilon}'_{\vec{k}\sigma}$ another way. In the K' frame, the distribution $n'_{\vec{k}\sigma} = n^0_{\vec{k}+\hbar^{-1}m\vec{u},\sigma}$ is shifted to a filled Fermi sphere centered at $\vec{k} = -m\vec{u}/\hbar$, as depicted in the figure below. Thus $n'_{\vec{k}\sigma} = n^0_{\vec{k}\sigma} + \hbar^{-1}m\vec{u}\cdot\vec{\nabla}_{\vec{k}}\, n^0_{\vec{k}\sigma}$, and the



energy $\tilde{\varepsilon}'_{\vec{k}\sigma}$ is obtained as follows:

$$\tilde{\varepsilon}'_{\vec{k}\sigma} = \varepsilon_{\vec{k}\sigma} + \frac{1}{V}\sum_{\vec{k}',\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta n'_{\vec{k}'\sigma'}$$

$$= \varepsilon_{\vec{k}\sigma} - 2mv_F\int\frac{d^3k'}{(2\pi)^3}\, f^s_{\vec{k},\vec{k}'}\,\vec{u}\cdot\vec{k}'\,\delta(\varepsilon_{\vec{k}'}-\mu)$$

$$= \varepsilon_{\vec{k}\sigma} - mv_F\, g(\varepsilon_F)\,\vec{u}\cdot\int\frac{d\hat{k}'}{4\pi}\,\vec{k}' f^s_{\vec{k},\vec{k}'_F}$$

We set $\vec{k} = \vec{k}_F$ to lie on the FS, whence

$$\tilde{\varepsilon}'_{k_F\hat{k},\sigma} = \varepsilon_{k_F\hat{k},\sigma} - mv_F\,\vec{u}\cdot\underbrace{\int\frac{d\hat{k}'}{4\pi}\,\hat{k}'\,F^s(\vartheta_{\hat{k},\hat{k}'})}_{\text{must lie along }\hat{k}}$$

$$= \varepsilon_{k_F\hat{k},\sigma} - mv_F\,\vec{u}\cdot\hat{k}\int\frac{d\hat{k}'}{4\pi}\,\hat{k}\cdot\hat{k}'\,F^s(\vartheta_{\hat{k},\hat{k}'})$$

$$= \varepsilon_{k_F\hat{k},\sigma} - \frac{1}{3}F^s_1\, mv_F\,\vec{u}\cdot\hat{k}$$

Therefore

$$\tilde{\varepsilon}'_{k_F\hat{k},\sigma} = \varepsilon_{k_F\hat{k},\sigma} - \frac{1}{3}F^s_1\, mv_F\,\vec{u}\cdot\hat{k}$$

$$= \varepsilon_{k_F\hat{k}+\hbar^{-1}m\vec{u},\sigma} - \hbar k_F \hat{k}\cdot\vec{u} - \frac{1}{2}m\vec{u}^2$$

$$= \varepsilon_{k_F\hat{k},\sigma} + \hbar^{-1}m\vec{u}\cdot\vec{\nabla}_{\vec{k}}\,\varepsilon_{\vec{k},\sigma}\Big|_{\vec{k}=k_F\hat{k}} - \hbar k_F\hat{k}\cdot\vec{u} - \frac{1}{2}m\vec{u}^2$$

$$= \varepsilon_{k_F\hat{k},\sigma} + (m-m^*)\,v_F\,\vec{u}\cdot\hat{k} - \frac{1}{2}m\vec{u}^2$$

To order $\vec{u}$ we then have

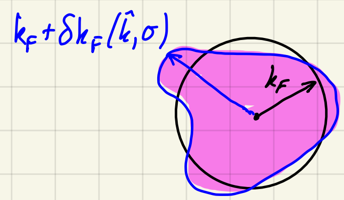
$$(m-m^*) = -\frac{1}{3}F_1^s\,m \quad\Rightarrow\quad \frac{m^*}{m} = 1+\frac{1}{3}F_1^s$$

This relation holds only in Galilean-invariant Fermi liquids.

- Thermodynamic stability at $T=0$

Consider a $T=0$ distortion of the FS, described by

$$n_{\vec{k}\sigma} = \Theta\big(k_F+\delta k_F(\hat{k},\sigma)-k\big)$$
$$= \Theta(k_F-k) + \delta(k_F-k)\,\delta k_F(\hat{k},\sigma) + \frac{1}{2}\delta'(k_F-k)\big[\delta k_F(\hat{k},\sigma)\big]^2 + \ldots$$



We now evaluate, with fixed N,

$$\Omega(T=0,V,\mu) = E-\mu N$$
$$= \Omega_0 + \sum_{\vec{k},\sigma}(\varepsilon_{\vec{k}\sigma}-\mu)\,\delta n_{\vec{k}\sigma} + \frac{1}{2V}\sum_{\vec{k},\sigma}\sum_{\vec{k}',\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta n_{\vec{k}\sigma}\,\delta n_{\vec{k}'\sigma'}$$

$$= \Omega_0 + \sum_{\vec{k},\sigma}(\varepsilon_{\vec{k}\sigma}-\mu)\Big\{\delta(k_F-k)\,\delta k_F(\hat{k},\sigma) + \frac{1}{2}\delta'(k_F-k)\big[\delta k_F(\hat{k},\sigma)\big]^2\Big\}$$
$$+ \frac{1}{2V}\sum_{\vec{k},\sigma}\sum_{\vec{k}',\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta(k_F-k)\,\delta(k_F-k')\,\delta k_F(\hat{k},\sigma)\,\delta k_F(\hat{k}',\sigma')$$

which yields

$$\frac{\Omega-\Omega_0}{V} = \frac{\hbar^2 k_F^3}{8\pi^2 m^*}\sum_{\nu=s,a}\left\{\int\frac{d\hat{k}}{4\pi}\big[\delta k_F^\nu(\hat{k})\big]^2 + \int\frac{d\hat{k}}{4\pi}\int\frac{d\hat{k}'}{4\pi}F^\nu(\vartheta_{\hat{k},\hat{k}'})\,\delta k_F^\nu(\hat{k})\,\delta k_F^\nu(\hat{k}')\right\}$$

where $\delta k_F^s(\hat{k}) = \sum_\sigma \delta k_F(\hat{k},\sigma)$ and $\delta k_F^a(\hat{k}) = \sum_\sigma \sigma\, \delta k_F(\hat{k},\sigma)$.

We now resolve the $\nu = s,a$ distortions into spherical harmonics, viz.

$$\delta k_F^\nu(\hat{k}) \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm}^\nu \, Y_{lm}(\hat{k})$$

with $A_{l,-m}^\nu(\hat{k}) = A_{lm}^{\nu *}(\hat{k})$. We also have

$$F^\nu(\vartheta_{\hat{k},\hat{k}'}) = \sum_{l=0}^{\infty} F_l^\nu \, P_l(\vartheta_{\hat{k},\hat{k}'}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} F_l^\nu \, Y_{lm}^*(\hat{k}) \, Y_{lm}(\hat{k}')$$

Thus, using the orthonormality of the Y_{lm}'s,

$$\int d\hat{k} \; Y_{lm}^*(\hat{k}) \, Y_{l'm'}(\hat{k}) = \delta_{ll'} \delta_{mm'}$$

we arrive at

$$\frac{\Omega - \Omega_0}{V} = \frac{\hbar^2 k_F^3}{32\pi^3 m^*} \sum_{\nu = s,a} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(1 + \frac{F_l^\nu}{2l+1}\right) \left| A_{lm}^\nu \right|^2$$

We see that there is a **stability criterion** for each FS distortion channel:

$$F_l^\nu > -(2l+1)$$

These relations must hold for all $\nu \in \{s,a\}$ and $l \in \{0,1,\ldots\}$ in order that the energy of the FL is lowest when the

FS is spherical. What happens when one or more of these channels goes unstable? In that case, the energy is unbounded from below, which is unphysical. We can repair this defect by introducing a phenomenological fourth order term in Ω,

$$\frac{\Delta\Omega}{V} = \frac{\hbar^2 k_F^3}{4\pi m^*} \sum_{\nu=s,a} \lambda_\nu \left(\int \frac{d\hat{k}}{4\pi} \left[\delta k_F^\nu(\hat{k},\sigma)\right]^2 \right)^2$$
$$= \frac{\hbar^2 k_F^3}{64\pi^3 m^*} \sum_{\nu=s,a} \lambda_\nu \left(\sum_{l,m} |A^\nu_{lm}|^2 \right)^2$$

with $\lambda_{s,a} > 0$. Then we have

$$\frac{\Omega + \Delta\Omega - \Omega_0}{V} = \frac{\hbar^2 k_F^3}{32\pi^3 m^*} \sum_{\nu=s,a} \left\{ \sum_{l,m} \left(1 + \frac{F^\nu_l}{2l+1}\right) |A_{lm}|^2 + \frac{1}{2}\lambda_\nu \left(\sum_{l,m} |A^\nu_{lm}|^2 \right)^2 \right\}$$

Now minimize wrt the amplitudes A^ν_{lm}. For stable channels where $F^\nu_l > -(2l+1)$, we get $A^\nu_{lm} = 0$. But for the unstable channels, we find

$$\sum_{m=-l}^{l} |A^\nu_{lm}|^2 = -\frac{1}{\lambda_\nu} \underbrace{\left(1 + \frac{F^\nu_l}{2l+1}\right)}_{<0} > 0$$

Thus an instability in the $l=1$ channel results in a dipolar FS distortion, $l=2 \Rightarrow$ quadrupolar distortion, etc.

More sophistocated model: