$$\frac{\Delta\Omega}{V} = \frac{\hbar^2 k_F^3}{64\pi^2 m^*} \sum_{\nu,\nu'} \sum_{l,l'} \Lambda^{\nu\nu'}_{ll'} \left(\sum_{m=-l}^{l} |A^{\nu}_{lm}|^2\right)\left(\sum_{m'=-l'}^{l'} |A^{\nu'}_{l'm'}|^2\right)$$

- **Lecture 11** (Feb. 9)
- Collective dynamics of the Fermi surface

Landau-Boltzmann equation:

$$\frac{\partial n_{k\sigma}}{\partial t} + \frac{1}{\hbar}\frac{\partial \tilde{\varepsilon}_{k\sigma}}{\partial k}\cdot\frac{\partial n_{k\sigma}}{\partial \vec{x}} - \frac{1}{\hbar}\frac{\partial \tilde{\varepsilon}_{k\sigma}}{\partial \vec{x}}\cdot\frac{\partial n_{k\sigma}}{\partial k} = I_{k\sigma}[n]$$

We include a local potential $V_\sigma(\vec{r},t)$, viz.

$$\tilde{\varepsilon}_{k\sigma}(\vec{x},t) = V_\sigma(\vec{x},t) + \varepsilon_{k\sigma} + \sum_{\sigma'}\int\frac{d^3k'}{(2\pi)^3} f_{k\sigma,k'\sigma'}\,\delta n_{k'\sigma'}(\vec{x},t)$$

Now linearize, writing $n_{k\sigma} = n^0_{k\sigma} + \delta n_{k\sigma}$:

$$\frac{\partial \delta n_{k\sigma}}{\partial t} + \frac{1}{\hbar}\frac{\partial \varepsilon_{k\sigma}}{\partial k}\cdot\frac{\partial \delta n_{k\sigma}}{\partial \vec{x}} - \frac{1}{\hbar}\frac{\partial n^0_{k\sigma}}{\partial k}\cdot\frac{\partial \tilde{\varepsilon}_{k\sigma}}{\partial \vec{x}} = (\mathcal{L}\,\delta n)_{k\sigma}$$

Here

$$\frac{\partial \tilde{\varepsilon}_{k\sigma}}{\partial \vec{x}} = \frac{\partial V_\sigma}{\partial \vec{x}} + \sum_{\sigma'}\int\frac{d^3k'}{(2\pi)^3} f_{k\sigma,k'\sigma'}\frac{\partial \delta n_{k'\sigma'}}{\partial \vec{x}}$$

If $V_\sigma(\vec{x},t) = \delta\hat{V}_\sigma\, e^{i(\vec{q}\cdot\vec{x}-\omega t)}$, then seek a solution with $\delta n_{k\sigma}(\vec{x},t) = \delta\hat{n}_{k\sigma}\, e^{i(\vec{q}\cdot\vec{x}-\omega t)}$, whence

$$\omega\,\delta\hat{n}_{k\sigma} - \vec{q}\cdot\vec{v}_{k\sigma}\,\delta\hat{n}_{k\sigma} + \left(\frac{\partial n^0_{k\sigma}}{\partial \varepsilon_{k\sigma}}\right)\vec{q}\cdot\vec{v}_{k\sigma}\left[\delta\hat{V}_\sigma + \sum_{\sigma'}\int\frac{d^3k'}{(2\pi)^3} f_{k\sigma,k'\sigma'}\,\delta n_{k'\sigma'}\right]$$

$$= -\left[\mathcal{L}\,\delta\hat{n}\right]_{k\sigma}$$

where $\mathcal{L}$ is the linearized collision operator.

Zero sound: Examine unforced, collisionless limit $\Rightarrow$

$$(\omega-\vec{q}\cdot\vec{v}_{\vec{k}\sigma})\,\delta\hat{n}_{\vec{k}\sigma}+\vec{q}\cdot\vec{v}_{\vec{k}\sigma}\left(\frac{\partial n^0_{\vec{k}\sigma}}{\partial\varepsilon_{\vec{k}\sigma}}\right)\sum_{\sigma'}\int\frac{d^3k'}{(2\pi)^3}\,f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta\hat{n}_{\vec{k}'\sigma'}=0$$

This is an eigenvalue equation for $\omega(\vec{q})$, where the eigenvector is $\delta\hat{n}_{\vec{q}\sigma}$. Assuming a small variation in the shape of the FS, write

$$\delta\hat{n}_{\vec{k}\sigma}=\hbar v_F\,\delta(\varepsilon_F-\varepsilon_{\vec{k}\sigma})\,\delta k_F(\hat{k},\sigma)$$

so that we obtain

$$(\omega-\vec{q}\cdot\vec{v}_{\vec{k}\sigma})\,\delta k_F(\hat{k},\sigma)-\vec{q}\cdot\vec{v}_{\vec{k}\sigma}\left(\frac{\partial n^0}{\partial\varepsilon}\right)_{\varepsilon_{\vec{k}\sigma}}\sum_{\sigma'}\int\frac{d^3k'}{(2\pi)^3}\,\delta(\varepsilon_F-\varepsilon_{\vec{k}'\sigma'})\,f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta k_F(\hat{k}',\sigma')=0$$

Taking $v_{\vec{k}\sigma}=v_F\hat{k}$, we arrive at

$$(\lambda-\vec{q}\cdot\hat{k})\,\delta k_F(\hat{k},\sigma)-\frac{1}{2}\vec{q}\cdot\hat{k}\int\frac{d\hat{k}'}{4\pi}\,F_{\vec{k}_F\sigma,\vec{k}'_F\sigma'}\,\delta k_F(\hat{k}',\sigma')=0$$

where $\lambda\equiv\omega/v_F q$. We can resolve this into the familiar symmetric and antisymmetric spin channels $\delta k_F^{s,a}(\hat{k})$, so

$$\delta k_F^{\nu}(\hat{k})=\frac{\hat{q}\cdot\hat{k}}{\lambda-\hat{q}\cdot\hat{k}}\int\frac{d\hat{k}'}{4\pi}\,F^{\nu}(\vartheta_{\hat{k},\hat{k}'})\,\delta k_F^{\nu}(\hat{k}')$$

and further writing

$$\delta k_F^{\nu}(\hat{k})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A^{\nu}_{lm}\,Y_{lm}(\hat{k})\quad,\quad F^{\nu}(\vartheta_{\hat{k},\hat{k}'})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi F_l^{\nu}}{2l+1}\,Y_{lm}(\hat{k})\,Y^{*}_{lm}(\hat{k}')$$

we find — given $\hat{q}$, solve for the eigenvalue $\lambda(\hat{q})$, eigenvector $A^\nu_{\ell m}(\hat{q})$

$$A^\nu_{\ell m} = \sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'} \underbrace{\frac{F^\nu_{\ell'}}{2\ell'+1}\left[\int d\hat{k}\,\frac{\hat{q}\cdot\hat{k}}{\lambda-\hat{q}\cdot\hat{k}}\,Y^*_{\ell m}(\hat{k})Y_{\ell' m'}(\hat{k})\right]}_{\equiv R_{\ell m,\ell' m'}(\lambda,\hat{q})} A^\nu_{\ell' m'}$$

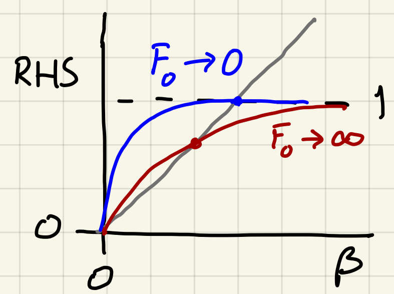
This is still an eigenvalue equation, where λ is the eigenvalue and $A^\nu_{\ell m}$ the eigenvector.

Simple model for zero sound: take $F^\nu_\ell = F^\nu_0\,\delta_{\ell,0}$. Then, dropping the ν label,

$$1 = F_0\int\frac{d\hat{k}}{4\pi}\,\frac{\hat{q}\cdot\hat{k}}{\lambda-\hat{q}\cdot\hat{k}} = F_0\left\{\frac{\lambda}{2}\ln\left(\frac{\lambda+1}{\lambda-1}\right)-1\right\}$$

Equivalently,

$$\beta = \tanh\left[(1+F_0^{-1})\beta\right]$$



with $\beta\equiv\lambda^{-1}$. This is a transcendental equation for $\beta(F_0)$. A nontrivial solution exists provided $F_0>0$, with real positive β.

$$F_0\to 0^+: \qquad \beta \simeq 1-2e^{-2/F_0}$$

$$F_0\to\infty: \qquad \beta\simeq\sqrt{\frac{3}{F_0}}$$

For $F_0\in[-1,0]$ a solution with complex β exists, which corresponds to a damped oscillation.

Dynamical response of the Fermi liquid:

We now restore the driving term $\delta\hat{V}(\vec{q},\omega)\, e^{i(\vec{q}\cdot\vec{x}-\omega t)}$. Again we work in the collisionless limit, where we have

$$\delta k_F^s(\hat{k}) = \frac{\hat{q}\cdot\hat{k}}{\lambda-\hat{q}\cdot\hat{k}}\left\{\int\frac{d\hat{k}'}{4\pi}\, F^s(\vartheta_{\hat{k},\hat{k}'})\,\delta k_F^s(\hat{k}') + \frac{\delta\hat{V}(\vec{q},\omega)}{\hbar v_F}\right\}$$

with $\lambda = \omega/q v_F$ as before. The density response is related to the FS distortion by

$$\delta\hat{n}(\vec{q},\omega) = \frac{k_F^2}{\pi^2}\int\frac{d\hat{k}}{4\pi}\,\delta k_F^s(\hat{k})$$

Note that $\delta k_F^s(\hat{k})$ is implicitly a function of ω as well. Unfortunately, the various angular momentum channels no longer decouple. Still, we may make progress if we assume $F^s(\vartheta) = F_0^s$ is isotropic, in which case

$$\delta\hat{n}(\vec{q},\omega) = \int\frac{d\hat{k}}{4\pi}\,\frac{\hat{q}\cdot\hat{k}}{\lambda-\hat{q}\cdot\hat{k}}\left\{F_0\,\delta\hat{n}(\vec{q},\omega) + \frac{k_F^2}{\pi^2}\,\frac{\delta\hat{V}(\vec{q},\omega)}{\hbar v_F}\right\}$$

We may then solve for the susceptibility,

$$\hat{\chi}(\vec{q},\omega) \equiv -\frac{\delta\hat{n}(\vec{q},\omega)}{\delta\hat{V}(\vec{q},\omega)} = \frac{g(\varepsilon_F)\, G(\omega/v_F q)}{1+F_0\, G(\omega/v_F q)}$$

where

$$G(\lambda) = -\int\frac{d\hat{k}}{4\pi}\,\frac{\hat{q}\cdot\hat{k}}{\lambda-\hat{q}\cdot\hat{k}} = 1 - \frac{\lambda}{2}\ln\left(\frac{\lambda+1}{\lambda-1}\right)$$

The poles of the response function $\hat{\chi}(\vec{q},\omega)$ correspond to solutions of $1+F_0\,G(\omega/v_F q)=0$, which is the equation for zero sound. Thus, as is familiar from the elementary physics of driven oscillators, the response diverges when the driving frequency matches the natural frequency.

- **Linear response of quantum systems**

We have already described the linear response of systems described by DFT and by FLT. Here we consider the general context. Let our Hamiltonian be $\hat{H}=\hat{H}_0+\hat{H}_1(t)$, where

$$\hat{H}_1(t) = -\sum_i \phi_i(t)\,\hat{Q}_i$$

Here $\{\hat{Q}_i\}$ are operators and $\{\phi_i(t)\}$ are time-dependent fields. Examples:

$\hat{Q}_i$	$\phi_i(t)$
$-\hat{M}^\alpha$	$B^\alpha(t)$
$\hat{\rho}(\vec{x})$	$\phi(x,t)$
$\vec{j}(\vec{x})$	$-\frac{1}{c}\vec{A}(\vec{x},t)$

We may subsume spatial labels $\vec{x}$ in the operator index i. The wavefunction $|\Psi(t)\rangle$ evolves according to the

Schrödinger equation $i\hbar\,\partial_t|\Psi(t)\rangle = \hat{H}(t)\,|\Psi(t)\rangle$.
We assume $\langle\Psi_0|\hat{Q}_i|\Psi_0\rangle = 0$, where $|\Psi_0\rangle$ is the ground state of $\hat{H}_0$. The time-dependent expectation of $\hat{Q}_i$ is given by

$$Q_i(t) = \langle\Psi(t)|\,\hat{Q}_i\,|\Psi(t)\rangle$$
$$\equiv \int_{-\infty}^{\infty} dt'\,\chi_{ij}(t-t')\,\phi_j(t') + O(\phi^2)$$

where $\chi_{ij}(t-t')$ is a **response function** describing how $Q_i(t)$ depends on $\phi_j(t')$. Thus we may write

$$\chi_{ij}(t-t') = \left.\frac{\delta Q_i(t)}{\delta\phi_j(t')}\right|_{\vec{\phi}=0}$$

Note that $\chi_{ij}(t-t') = 0$ for $t<t'$, which is a reflection of **causality**. Before deriving a general expression for $\chi_{ij}(t-t')$ we pause to discuss the important consequences of causality.

Causality and Kramers–Kronig relations:

Let's drop the i and j indices and consider the case

$$x(t) = \int_{-\infty}^{\infty} dt'\,\chi(t-t')\,f(t') \qquad (*)$$

For example, we could have a forced harmonic oscillator,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t)$$

The solution is of the form (*) plus an arbitrary solution to the homogeneous equation $\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0$.

Recall that the Fourier transform of a convolution is the product of Fourier transforms, hence

$$\hat{x}(\omega) = \hat{\chi}(\omega)\, \hat{f}(\omega)$$

where

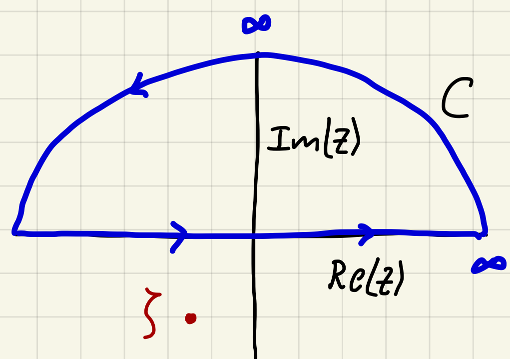
$$\hat{\chi}(\omega) = \int_{-\infty}^{\infty} ds\, \chi(s)\, e^{i\omega s} \quad , \quad \chi(s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat{\chi}(\omega)\, e^{-i\omega s}$$

With $s<0$, we can close the integral for $\chi(s)$ in the UHP, hence

$$\chi(s<0) = 0 \quad \Rightarrow \quad \hat{\chi}(z) \text{ analytic in UHP}$$

This means

$$\oint_C \frac{dz}{2\pi i}\, \frac{\hat{\chi}(z)}{z - \zeta} = 0$$



provided $\mathrm{Im}(\zeta) < 0$. For $\omega \in \mathbb{R}$, define

$$\hat{\chi}(\omega) \equiv \lim_{\epsilon \to 0^+} \hat{\chi}(\omega + i\epsilon) \equiv \hat{\chi}'(\omega) + i\, \hat{\chi}''(\omega)$$

I.e.

$$\hat{\chi}'(\omega) \equiv \mathrm{Re}\, \hat{\chi}(\omega) \quad , \quad \hat{\chi}''(\omega) \equiv \mathrm{Im}\, \hat{\chi}(\omega)$$

Assuming $\hat{\chi}(z)$ vanishes sufficiently rapidly along the arc of the contour C that Jordan's lemma applies, we have

$$0 = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i} \, \frac{\hat{\chi}(\nu)}{\nu - \omega + i\epsilon} = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i} \left[\hat{\chi}'(\nu) + i\, \hat{\chi}''(\nu) \right] \left[\frac{\mathcal{P}}{\nu - \omega} - i\pi\, \delta(\nu - \omega) \right]$$

Taking the real and imaginary parts of the above equation yield the **Kramers-Kronig relations**,

$$\hat{\chi}'(\omega) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \, \frac{\hat{\chi}''(\nu)}{\nu - \omega} \quad , \qquad \hat{\chi}''(\omega) = -\mathcal{P} \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \, \frac{\hat{\chi}'(\nu)}{\nu - \omega}$$

Note that

$$\frac{1}{x + i\epsilon} = \frac{x}{x^2 + \epsilon^2} - \frac{i\epsilon}{x^2 + \epsilon^2} = \mathcal{P} \, \frac{1}{x} - i\pi\, \delta(x)$$

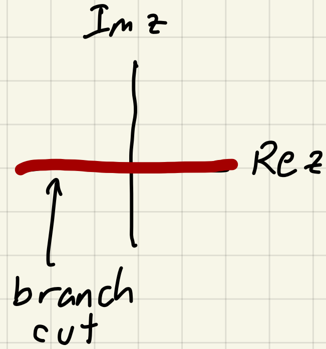
since $\int_{-\infty}^{\infty} dx \, \frac{\epsilon}{x^2 + \epsilon^2} = \pi$ and $\mathcal{P} \int_{-\infty}^{\infty} dx \, \frac{f(x)}{x} = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right] dx \, \frac{f(x)}{x}$

Furthermore, we may analytically continue $\hat{\chi}(\nu)$ off the $\nu \in \mathbb{R}$ axis, writing

$$\hat{\chi}(z) = \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \, \frac{\hat{\chi}''(\nu)}{\nu - z} = -i \, \mathrm{sgn}(\mathrm{Im}\, z) \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \, \frac{\hat{\chi}'(\nu)}{\nu - z}$$

This guarantees the result

$$\lim_{\epsilon \to 0} \left\{ \hat{\chi}(\omega + i\epsilon) - \hat{\chi}(\omega - i\epsilon) \right\} = 2i\, \hat{\chi}''(\omega)$$



Example: Suppose $\hat{\chi}''(\nu) = \frac{\nu}{\nu^2 + \gamma^2}$. Then

$$\hat{\chi}'(\nu) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\zeta}{\pi} \, \frac{\zeta}{\zeta^2 + \gamma^2} \, \frac{1}{\zeta - \nu} = \frac{\gamma}{\nu^2 + \gamma^2}$$

Thus $\hat{\chi}'(\nu) + i\hat{\chi}''(\nu) = \dfrac{1}{\gamma - i\nu} = \dfrac{i}{\nu + i\gamma}$ can be continued to an analytic function in the UHP. Note

$$\hat{\chi}(z) = \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \frac{\nu}{\nu^2+\gamma^2} \frac{1}{\nu - z} = \begin{cases} +i/(z+i\gamma) & \text{if } \mathrm{Im}\, z > 0 \\ -i/(z-i\gamma) & \text{if } \mathrm{Im}\, z < 0 \end{cases}$$

Thus there is a **branch cut** everywhere along the $\mathrm{Re}\, z$ axis, with $\hat{\chi}(\omega \pm i\epsilon) = \pm i/(\omega \pm i\gamma)$. If instead we analytically continue $\hat{\chi}(z)$ from the UHP into the LHP, we obtain $\hat{\chi}_{AC}(z) = i/(z+i\gamma)$ which has a simple pole at $z = -i\gamma$ and no branch cut

Explicit expression for $\hat{\chi}_{ij}(\omega)$:

Recall $\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$ and $i\hbar\, \partial_t |\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle$. Formally integrating, we have

$$|\Psi(t)\rangle = \hat{T} \exp\left\{ -\frac{i}{\hbar} \int_{t_0}^{t} dt'\, \hat{H}(t') \right\} |\Psi(t_0)\rangle$$

time ordering operator, places earliest times to right

$$= \lim_{N\to\infty} \left(1 - \frac{i\epsilon}{\hbar} \hat{H}(t_0 + (N-1)\epsilon)\right) \cdots \left(1 - \frac{i\epsilon}{\hbar} \hat{H}(t_0)\right) |\Psi(t_0)\rangle$$

with $N\epsilon \equiv t - t_0$ fixed. Thus,

$$\hat{U}(t_2, t_1) = \lim_{N\to\infty} \left(1 - \frac{i\epsilon}{\hbar} \hat{H}(t_0 + (N-1)\epsilon)\right) \cdots \left(1 - \frac{i\epsilon}{\hbar} \hat{H}(t_0)\right)$$

is the **time-evolution operator**, which satisfies

Scratch

$$\ddot{x} + \omega_0^2(t)\, x = 0$$

$$\frac{dx}{dt} = v \quad , \quad \frac{dv}{dt} = -\omega_0^2(t)\, x$$

$$\frac{d}{dt}\underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{\vec{\varphi}(t)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_0^2(t) & 0 \end{pmatrix}}_{M(t)}\begin{pmatrix} x \\ v \end{pmatrix}$$

$$\dot{\vec{\varphi}}(t) = M(t)\,\vec{\varphi}(t)$$

$$\vec{\varphi}(t) = \underbrace{T \exp\left\{\int_{t_0}^{t} dt'\, M(t')\right\}}_{= U(t,t_0)} \vec{\varphi}(t_0)$$

$$= \lim_{\substack{N\to\infty \\ N\epsilon = t-t_0}} \underbrace{\Big(1 + \epsilon M(t_0 + (N-1)\epsilon)\Big)\cdots\Big(1+\epsilon M(t_0)\Big)}_{= U(t,t_0)} \vec{\varphi}(t_0)$$

If $[M(t), M(t')] = 0$, then

$$U(t,t_0) = T\exp\left\{\int_{t_0}^{t} dt' M(t')\right\} = \exp\left\{\int_{t_0}^{t} dt' M(t')\right\}$$

Example: $M(t) = a(t)\,\mathbb{1} + b(t)\,\sigma^z$

the composition rule

$$\hat{U}(t_3, t_1) = \hat{U}(t_3, t_2)\,\hat{U}(t_2, t_1)$$

for any $t_1 < t_2 < t_3$. The solution to the Schrödinger equation satisfies $|\Psi(t_2)\rangle = \hat{U}(t_2, t_1)\,|\Psi(t_1)\rangle$. If $t_1 < t < t_2$, we may functionally differentiate

$$\frac{\delta \hat{U}(t_2, t_1)}{\delta \phi_j(t)} = \frac{i}{\hbar}\,\hat{U}(t_2, t)\,\hat{Q}_j\,\hat{U}(t, t_1)$$

Our aim, recall, is to compute the response function

$$\chi_{ij}(t - t') = \frac{\delta \langle \Psi(t)|\hat{Q}_i|\Psi(t)\rangle}{\delta \phi_j(t')}$$

To this end, note that

$$\left.\frac{\delta|\Psi(t)\rangle}{\delta \phi_j(t')}\right|_{\vec{\phi}=0} = \frac{i}{\hbar}\,\underbrace{e^{-i\hat{H}_0(t-t')/\hbar}}_{\hat{U}_0(t,t')}\,\hat{Q}_j\,\underbrace{e^{-i\hat{H}_0(t'-t_0)/\hbar}}_{\hat{U}_0(t',t_0)}\,|\Psi(t_0)\rangle\,\Theta(t-t')$$

$$= \frac{i}{\hbar}\,e^{-i\hat{H}_0 t/\hbar}\,\hat{Q}_j(t')\,e^{+i\hat{H}_0 t_0/\hbar}\,|\Psi(t_0)\rangle\,\Theta(t-t')$$

where $\hat{Q}_j(t') = e^{i\hat{H}_0 t'/\hbar}\,\hat{Q}_j\,e^{-i\hat{H}_0 t'/\hbar}$ is the operator $\hat{Q}_j$ in the **interaction representation**. We now have

$$\chi_{ij}(t-t') = \frac{\delta \langle \Psi(t)|}{\delta \phi_j(t')}\,\hat{Q}_i\,|\Psi(t)\rangle + \langle \Psi(t)|\,\hat{Q}_i\,\frac{\delta|\Psi(t)\rangle}{\delta \phi_j(t')}$$

$$= \Big\{ -\frac{i}{\hbar}\,\langle \Psi_0|e^{-i\hat{H}_0 t_0/\hbar}\,\hat{Q}_j(t')\,e^{+i\hat{H}_0 t/\hbar}\,\hat{Q}_i\,e^{-i\hat{H}_0 t/\hbar}\,e^{+i\hat{H} t_0/\hbar}\,|\Psi_0\rangle$$

$$+\frac{i}{\hbar}\langle\Psi_0|e^{-i\hat{H}_0 t/\hbar}e^{i\hat{H}_0 t/\hbar}\hat{Q}_i e^{-i\hat{H}_0 t/\hbar}\hat{Q}_j(t')e^{i\hat{H}_0 t_0/\hbar}|\Psi_0\rangle\Big\}\times\Theta(t-t')$$

$$=\frac{i}{\hbar}\langle[\hat{Q}_i(t),\hat{Q}_j(t')]\rangle\,\Theta(t-t')$$

where $\langle\cdots\rangle$ is taken in the state $|\tilde{\Psi}_0\rangle = e^{-i\hat{H}_0 t_0/\hbar}|\Psi_0\rangle$ and where $t'>t_0$ by assumption. We may take $t_0\to-\infty$. Thus we have obtained the important result

$$\chi_{ij}(t-t') = \frac{i}{\hbar}\langle[\hat{Q}_i(t),\hat{Q}_j(t')]\rangle\,\Theta(t-t')$$

The average $\langle\cdots\rangle$ may also be taken with respect to a Gibbs-weighted distribution of initial states, for the case when $T>0$.

- **Lecture 12** (Feb. 11) **Spectral representation:**

Inserting a resolution of the identity $\hat{I}=\sum_n |n\rangle\langle n|$, where $\hat{H}_0|n\rangle = E_n^0|n\rangle$, we have the **spectral representation** of the response function,

$$\hat{\chi}_{ij}(\omega+i\epsilon) = \frac{i}{\hbar}\int_0^\infty dt\,\langle[\hat{Q}_i(t),\hat{Q}_j(0)]\rangle\,e^{i\omega t}e^{-\epsilon t}$$

$$=\frac{1}{\hbar}\sum_{m,n}P_m\left\{\frac{\langle m|\hat{Q}_j|n\rangle\langle n|\hat{Q}_i|m\rangle}{\omega-\omega_m+\omega_n+i\epsilon}-\frac{\langle m|\hat{Q}_i|n\rangle\langle n|\hat{Q}_j|m\rangle}{\omega+\omega_m-\omega_n+i\epsilon}\right\}$$

where $\omega_m \equiv (E_m^0-E_0^0)/\hbar$ is the m^{th} excitation frequency. This is often called the **retarded response function**, because of the $\Theta(t-t')$ factor in $\chi_{ij}(t-t')$. A related