$$F = F_0 + \int d^3x \left\{ \frac{\vec{B}^2}{8\pi} + \frac{\lambda_L^2}{8\pi} (\vec{\nabla}\times\vec{B})^2 \right\}$$

whence

$$\frac{\delta F}{\delta \vec{B}(\vec{x})} = 0 \Rightarrow \vec{B} - \lambda_L^2 \vec{\nabla}\times\vec{B} = 0$$

- Ginzburg – Landau theory

In ^{4}He, the order parameter is $\Psi(\vec{x}) = \langle \underbrace{\psi}_{\text{field operator (bosonic)}}(\vec{x}) \rangle$.

$\Psi \neq 0 \Leftrightarrow$ Bose – Einstein condensation. Fermions cannot condense!

Rather, the order parameter of an s-wave superconductor is

$$\Psi(\vec{x}) = \langle \underbrace{\psi_\uparrow(\vec{x})\,\psi_\downarrow(\vec{x})}_{\substack{\text{composite operator} \\ \text{with BE statistics}}} \rangle$$

Lecture 14 (Feb. 18)

Landau theory: The SC order parameter $\Psi(\vec{x})$ is a complex scalar. Assuming a homogeneous $\Psi(\vec{x}) = \text{const.}$, we write the Landau free energy as an expansion in powers of Ψ, viz.

$$f(\Psi, \Psi^*) = a|\Psi|^2 + \tfrac{1}{2} b |\Psi|^4$$

with $a, b \in \mathbb{R}$ and $b > 0$ for stability. The free energy has an O(2) symmetry under $\Psi \to e^{i\alpha}\Psi$, $\Psi^* \to e^{-i\alpha}\Psi^*$.

Minimizing f, we find

$$\Psi = \begin{cases} \sqrt{-a/b}\, e^{i\varphi} & , \text{ if } a < 0 \\ 0 & , \text{ if } a > 0 \end{cases}$$

When $a<0$, the order parameter chooses a direction $e^{i\varphi}$ which spontaneously breaks $O(2)$. Recall $a(T) = \alpha(T-T_c)$ near $T=T_c$ so $T<T_c$ is the ordered phase. The free energy is then

$$f = \begin{cases} -a^2/2b & \text{if } a<0 \\ 0 & \text{if } a>0 \end{cases}$$

and since $f_s - f_n = -\frac{1}{8\pi} H_c^2(T)$, we identify

$$\frac{a^2(T)}{b(T)} = \frac{H_c^2(T)}{4\pi}$$

From London theory, $\lambda_L^2 = mc^2/4\pi n_s e^2$, so if we normalize $|\Psi|^2 = n_s/n$ then

$$|\Psi(T)|^2 = \frac{\lambda_L^2(0)}{\lambda_L^2(T)} = -\frac{a(T)}{b(T)}$$

Combined, these results yield

$$a(T) = -\frac{H_c^2(T)}{4\pi} \cdot \frac{\lambda_L^2(T)}{\lambda_L^2(0)} \quad , \quad b(T) = \frac{H_c^2}{4\pi} \cdot \frac{\lambda_L^4(T)}{\lambda_L^4(0)}$$

in the superconducting phase. Close to the transition, $H_c(T)$ vanishes in proportion to $\lambda_L^{-2}(T)$, so $a(T_c)=0$, while $b(T_c)>0$. We have for $c = -T\,\partial^2 f/\partial T^2$

$$\Delta c = c_s(T_c) - c_n(T_c) = \frac{T_c\,[a'(T_c)]^2}{b(T_c)}$$

In the vicinity of T_c, we may write $a(T) \approx a'(T_c)(T - T_c)$.

Ginzburg-Landau theory: Gradients in the order parameter must cost energy, so write

$$f = a|\Psi|^2 + \frac{1}{2}b|\Psi|^4 + K|\vec{\nabla}\Psi|^2 + \dots$$

From K we can derive a length scale, $\xi \equiv \sqrt{K/|a|}$, which is the **coherence length.** Since superconductors are charged, we extend the above using minimal coupling, viz.

$$f = a|\Psi|^2 + \frac{1}{2}b|\Psi|^4 + K\left|\left(\vec{\nabla} + \frac{ie^*}{\hbar c}\vec{A}\right)\Psi\right|^2 + \frac{1}{8\pi}(\vec{\nabla}\times\vec{A})^2$$

Here $e^* = 2e$ = condensate charge. Under a local gauge transformation, $\vec{A} \to \vec{A} - \frac{\hbar c}{e^*}\vec{\nabla}\alpha$ and $\Psi \to e^{i\alpha}\Psi$. Since gauge transformations result in no physical consequences, we conclude that **longitudinal** phase fluctuations of a charged system's order parameter don't physically exist.

Equations of motion: With $F = \int d^3x\, f(\Psi, \Psi^*, \vec{\nabla}\Psi, \vec{\nabla}\Psi^*)$ we compute

① $$\frac{\delta F}{\delta \Psi^*} = a\Psi + b|\Psi|^2\Psi - K\left(\vec{\nabla} + \frac{ie^*}{\hbar c}\vec{A}\right)^2\Psi$$

② $$\frac{\delta F}{\delta \vec{A}} = \frac{2Ke^*}{\hbar c}\left\{\frac{1}{2i}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) + \frac{e^*}{\hbar c}|\Psi|^2\vec{A}\right\} + \frac{\vec{\nabla}\times\vec{B}}{4\pi}$$

The second of these is the Ampère-Maxwell law, $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$, with

$$\vec{j} = -c\,\frac{\delta F_{matter}}{\delta \vec{A}} = -\frac{2Ke^*}{\hbar^2}\left\{\frac{\hbar}{2i}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) + \frac{e^*}{c}|\Psi|^2\vec{A}\right\}$$

When $\Psi = \text{const.}$, we then have $\vec{j} = -\frac{2Ke^{*2}}{\hbar^2 c}|\Psi|^2\vec{A}$ and taking the curl again yields

$$-\nabla^2\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c}\vec{\nabla}\times\vec{j} = -8\pi K\left(\frac{e^*}{\hbar c}\right)^2|\Psi|^2\,\vec{B} = -\lambda_L^{-2}\vec{B}$$

with

$$\lambda_L^{-2} = 8\pi K\left(\frac{e^*}{\hbar c}\right)^2|\Psi|^2 = \frac{8\pi a^2}{b}\cdot\frac{K}{|a|}\cdot\left(\frac{e^*}{\hbar c}\right)^2$$

since $|\Psi|^2 = -a/b$. Now from our previous results, we have that $a^2/b = H_c^2/4\pi$, thus

$$\lambda_L^{-2} = 2H_c^2\xi^2\left(\frac{e^*}{\hbar c}\right)^2 \Rightarrow H_c = \frac{\phi_L}{\sqrt{8}\,\pi\,\xi\lambda_L}$$

Critical current: Let $\Psi = \Psi_0 = \text{const.}$ Then

$$f = a|\Psi_0|^2 + \frac{1}{2}b|\Psi_0|^4 + K\left(\frac{e^*}{\hbar c}\right)^2\vec{A}^2|\Psi_0|^2$$

Consider $a<0$, i.e. $T<T_c$. Minimizing wrt $|\Psi_0|^2$ gives

$$|\Psi_0|^2 = \frac{|a| - K(e^*/\hbar c)^2\vec{A}^2}{b} > 0$$

and

$$\vec{j} = -2Kc\left(\frac{e^*}{\hbar c}\right)^2\left(\frac{|a| - K(e^*/\hbar c)^2\vec{A}^2}{b}\right)\vec{A}$$

In other words, in Cartesian coordinates,

$$\lambda_L^2 \nabla^2 \vec{B} = \vec{B} + \frac{\phi_L}{2\pi} \vec{\nabla} \times \vec{\nabla} \varphi$$

Normally $\vec{\nabla} \times \vec{\nabla}\varphi = 0$. However this fails when φ is not single valued! Assume $\vec{B} = B\hat{z}$. Then φ has singularities in the form of **vortex lines**. Write $\vec{x} = (\vec{\rho}, z)$, whence

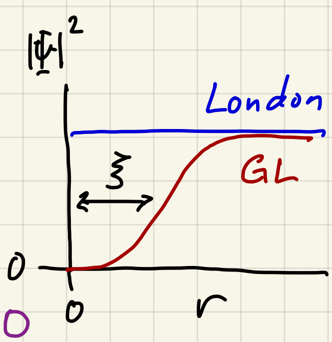
$$\lambda_L^2 \nabla^2 B(\vec{\rho}) = B(\vec{\rho}) + \phi_L \sum_i n_i \, \delta(\vec{\rho} - \vec{\rho}_i)$$

where $n_i \in \mathbb{Z}$ is the dimensionless quantized vorticity of the i^{th} singularity. Here we assume no spatial variations along $\hat{z}$. To solve, take the Fourier transform:

$$\hat{B}(\vec{q}) = -\frac{\phi_L}{1 + \lambda_L^2 \vec{q}^{\,2}} \sum_i n_i \, e^{-i\vec{q}\cdot\vec{\rho}_i}$$

McDonald function

$$\Rightarrow \quad B(\vec{\rho}) = -\frac{\phi_L}{2\pi \lambda_L^2} \sum_i n_i K_0\!\left(\frac{|\vec{\rho} - \vec{\rho}_i|}{\lambda_L}\right)$$



Limits of $K_0(z)$:

$$K_0(z) = \begin{cases} -C - \ln(z/2) & \text{as} \quad z \to 0 \\ (\pi/2z)^{1/2} e^{-z} & \text{as} \quad |z| \to \infty \end{cases}$$

where $C = 0.5772166\ldots$ is the Euler constant. The log divergence as $\rho \to 0$ is an artifact of the London limit, in which the vortex core size goes to zero. Better to impose a smooth cutoff on a scale ξ. The current density for a single vortex at the origin is then

$$\vec{j}(\vec{x}) = \frac{nc}{4\pi} \vec{\nabla} \times \vec{B} = -\frac{c}{4\pi\lambda_L^2} \cdot \frac{\phi_L}{2\pi\lambda_L^2} K_1(\rho/\lambda_L)\,\hat{\varphi}$$

with $K_1(z) = -K_0'(z)$. The total magnetic flux carried by the i^{th} vortex is $n_i \phi_L$.

Domain walls: Let's take $\vec{B} = 0$ and set $\vec{A} = 0$ everywhere, and consider the equation

$$\frac{\delta F}{\delta \Psi^*(\vec{x})} = a\,\Psi(\vec{x}) + b\,|\Psi(\vec{x})|^2\,\Psi(\vec{x}) - K\nabla^2\Psi(\vec{x}) = 0$$

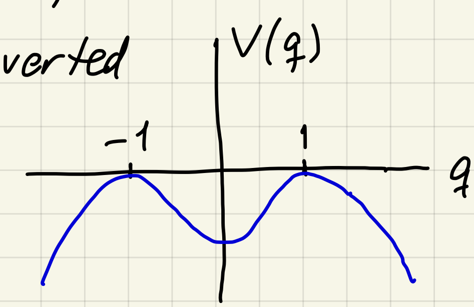
Lets scale, writing $\Psi = (|a|/b)^{1/2}\psi$, yielding

$$-\xi^2\nabla^2\psi + \mathrm{sgn}(T-T_c)\psi + |\psi|^2\psi = 0$$

Consider $T < T_c$. Let $\psi(\vec{x}) = f(x)\,e^{i\alpha}$ with α constant. So the only variation is along x. Thus,

$$-\xi^2 f'' - f + f^3 = 0 \quad \Rightarrow \quad \xi^2 \frac{d^2 f}{dx^2} = \frac{\partial}{\partial f}\left[\frac{1}{4}(1-f^2)^2\right]$$

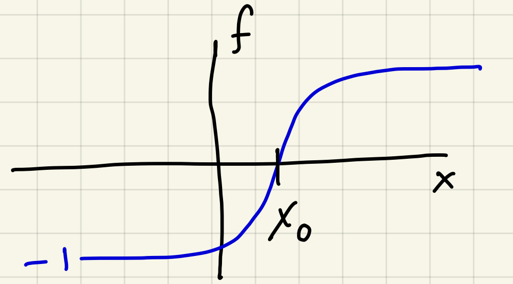
This looks just like $F = ma$, i.e. $m\ddot{q} = -V'(q)$, if we set $q = f$, $t = x$, $V(q) = -\frac{1}{4}(1-q^2)^2$, $m = \xi^2$. Familiar to us as motion in an inverted double well. Integrate once:



$$\xi^2\left(\frac{df}{dx}\right)^2 = \frac{1}{2}(1-f^2)^2 + C$$

Since $f(\infty) = 1$, we have $C = 0$. Now integrate a second time to obtain

$$f(x) = \tanh\left(\frac{x-x_0}{\sqrt{2}\,\xi}\right)$$



Thus, we interpolate between $f(0)=0$ and $f(\pm\infty)=\pm 1$ in a smooth fashion. This is called a **domain wall.**

The energy per unit length of the domain wall is

$$\tilde{\sigma} = \int_0^\infty dx \left\{ K \left|\frac{d\Psi}{dx}\right|^2 + a|\Psi|^2 + \frac{1}{2} b |\Psi|^4 \right\}$$

$$= \frac{a^2}{b} \int_0^\infty dx \left\{ \xi^2 \left(\frac{df}{dx}\right)^2 - f^2 + \frac{1}{2} f^4 \right\}$$

How does this compare with the energy of the bulk superconducting state? The difference is

$$\sigma = \tilde{\sigma} - \int_0^\infty dx \left(-\frac{H_c^2}{8\pi}\right)$$

$$= \frac{a^2}{b} \int_0^\infty dx \left\{ \xi^2 \left(\frac{df}{dx}\right)^2 + \frac{1}{2}(1-f^2)^2 \right\} \equiv \frac{H_c^2}{8\pi} \cdot \delta$$

Here

$$\delta = 2 \int_0^\infty dx \,(1-f^2) = \frac{4}{3}\sqrt{2}\,\xi$$

If we allowed a field to penetrate a distance λ_L in the DW state, we'd have obtained

$$\delta(T) = \frac{4}{3}\sqrt{2}\,\xi(T) - \lambda_L(T) \qquad \text{(approximation!)}$$

Detailed calculations show

$$\delta = \begin{cases} \frac{4}{3}\sqrt{2}\,\xi \approx 1.89\,\xi & \text{if } \xi \gg \lambda_L \\ 0 & \text{if } \xi = \sqrt{2}\,\lambda_L \\ -\frac{8}{3}(\sqrt{2}-1)\,\lambda_L & \text{if } \xi \ll \lambda_L \end{cases}$$

Recall $K \equiv \lambda_L/\xi$. So:

- **Type-I**: $K < \frac{1}{\sqrt{2}}$ and $\delta > 0$; surface energy prefers a spatially homogeneous sample for $T < T_c$
- **Type-II**: $K > \frac{1}{\sqrt{2}}$ and $\delta < 0$; negative surface energy causes the sample to break into domains, which are vortex solutions.

Applications of Ginzburg-Landau theory:

First, let's get rid of some constants by rescaling:

$$\Psi \equiv \sqrt{\frac{|a|}{b}}\,\psi \quad , \quad \vec{x} \equiv \lambda_L \vec{r} \quad , \quad \vec{A} \equiv \sqrt{2}\,\lambda_L H_c \vec{a} \quad , \quad \vec{H} = \sqrt{2}\,H_c \vec{h}$$

so that ψ, $\vec{r}$, and $\vec{a}$ are all dimensionless. Recall

$$K = \frac{\lambda_L}{\xi} = \frac{\sqrt{2}\,e^*}{\hbar c} H_c \lambda_L^2 = \sqrt{8}\,\pi \frac{H_c \lambda_L^2}{\phi_L}$$

Then we may write

$\lambda_L \vec{\nabla} \equiv \vec{\partial}$

$$G = \frac{H_c^2 \lambda_L^3}{4\pi} \int d^3r \left\{ -|\psi|^2 + \frac{1}{2}|\psi|^4 + |(K^{-1}\vec{\partial} + i\vec{a})\psi|^2 + (\vec{\partial} \times \vec{a})^2 - 2\vec{h} \cdot \vec{\partial} \times \vec{a} \right\}$$

Setting $\delta G = 0$, we obtain

$$① \quad (\kappa^{-1}\vec{\partial} + i\vec{a})^2\psi + \psi - |\psi|^2\psi = 0$$

$$② \quad \vec{\partial}\times(\vec{\partial}\times\vec{a} - \vec{h}) + |\psi|^2\vec{a} - \frac{i}{2\kappa}(\psi^*\vec{\partial}\psi - \psi\vec{\partial}\psi^*) = 0$$

In addition, we have the boundary condition

$$③ \quad \hat{n}\cdot(\vec{\partial} + i\kappa\vec{a})\psi\Big|_{\partial\Omega} = 0$$

We'll consider one application of GLT, to magnetic properties of type-II superconductors.

Consider the behavior when SC is just beginning to set in, so $|\psi| \ll 1$. In this case, ① gives

$$-(\kappa^{-1}\vec{\partial} + i\vec{a})^2\psi = \psi + \overbrace{\mathcal{O}(|\psi|^2\psi)}^{\text{neglect}}$$

② then gives

$$\vec{\partial}\times(\vec{b} - \vec{h}) = \mathcal{O}(|\psi|^2)$$

and so $\vec{b} = \vec{h} + \vec{\partial}\zeta$, but from free energy considerations we conclude $\zeta = 0$. Assume $\vec{b} = \vec{h} = b\hat{z}$ and choose a gauge

$$\vec{a} = -\tfrac{1}{2}by\,\hat{x} + \tfrac{1}{2}bx\,\hat{y}$$

Now define the operators

$$\pi_x = \frac{1}{i\kappa}\frac{\partial}{\partial x} - \frac{1}{2}by\,, \qquad \pi_y = \frac{1}{i\kappa}\frac{\partial}{\partial y} + \frac{1}{2}bx$$

which satisfy $[\pi_x, \pi_y] = b/iK$. Note that

$$-(K^{-1}\vec{\partial} + \vec{a})^2 = -\frac{1}{K^2}\frac{\partial^2}{\partial z^2} + \pi_x^2 + \pi_y^2$$

Ladder operators:

$$\gamma = \sqrt{\frac{K}{2b}}\,(\pi_x - i\pi_y) \quad , \quad \gamma^\dagger = \sqrt{\frac{K}{2b}}\,(\pi_x + i\pi_y)$$

with $[\gamma, \gamma^\dagger] = 0$. Then

$$\hat{L} \equiv -(K^{-1}\vec{\partial} + \vec{a})^2 = -\frac{1}{K^2}\frac{\partial^2}{\partial z^2} + \frac{2b}{K}\left(\gamma^\dagger\gamma + \frac{1}{2}\right)$$

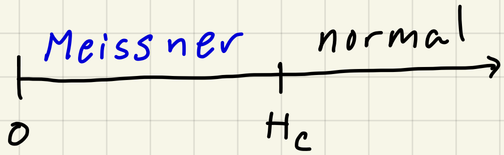
The lowest eigenvalue of $\hat{L}$ is then b/K, corresponding to $\gamma^\dagger\gamma = 0$. The full set of eigenvalues is given by

$$\mathcal{E}_n(k_z) = \frac{k_z^2}{K^2} + (2n+1)\frac{b}{K}$$

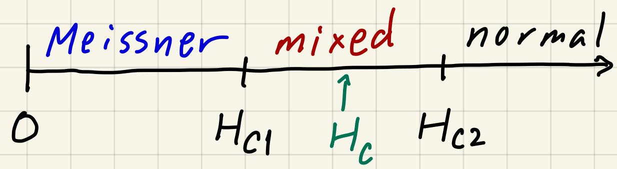
The lowest eigenvalue crosses the threshold of 1 when $b = h = K$, i.e. when $B = H = \sqrt{2}\,K H_c \equiv H_{c2}$.

Conclusion: If $H_{c2} < H_c = \frac{\phi_L}{\sqrt{8}\,\pi\xi\lambda_L}$ ("thermodynamic critical field") then a complete Meissner effect occurs when H is decreased below H_c. The order parameter ψ then jumps discontinuously, and the transition is first order. This is the case $K < 2^{-1/2}$. But if $H_{c2} > H_c$ and $K > 2^{-1/2}$, a complete Meissner effect can't occur for $H > H_c$, hence $H \in [H_c, H_{c2}]$ is a **mixed phase**.

type-I ($\kappa < 1/\sqrt{2}$)



type-II ($\kappa > 1/\sqrt{2}$)



Find $H_{c1} = \dfrac{H_c}{\sqrt{2}\kappa} \ln(2e^{-C}\kappa) = \dfrac{\ln(1.23\kappa)}{\sqrt{2}\kappa} H_c \quad (\kappa \gg 1)$

$H_{c2} = \sqrt{2}\kappa H_c$