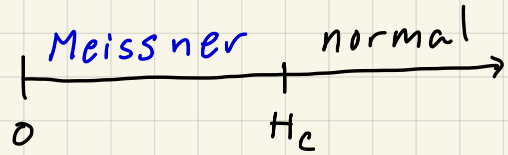
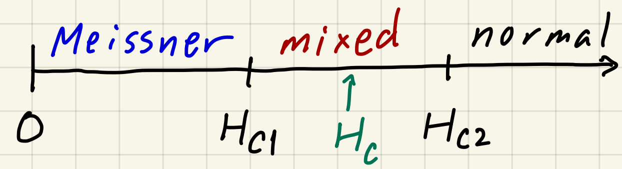
type – I $(\kappa < 1/\sqrt{2})$ type – II $(\kappa > 1/\sqrt{2})$





Find $H_{c1} = \dfrac{H_c}{\sqrt{2}\,\kappa} \ln\!\left(2e^{-C}\kappa\right) = \dfrac{\ln(1.123\,\kappa)}{\sqrt{2}\,\kappa} H_c \quad (\kappa \gg 1)$

$H_{c2} = \sqrt{2}\,\kappa\, H_c$; $H_c = \phi_L / \sqrt{8}\,\pi\xi\lambda_L = \dfrac{\kappa}{\sqrt{2}} \cdot \dfrac{\phi_L}{2\pi\lambda_L^2}$

Lecture 15 (Feb. 23)

Lower critical field of a type-II superconductor

We now ask: at what field H_{c1} do vortex lines first begin to penetrate a type-II superconductor? We assume $\rho \gg \xi$, i.e. the radial coordinate is large on the scale of the coherence length ξ, which is the vortex core size. In this limit we may assume $\psi \approx e^{i\varphi}$, whence the second GL eqn yields

$$\vec{\partial} \times \vec{b} = -\left(\kappa^{-1} \vec{\partial}\varphi + \vec{a}\right)$$

and the free energy in the presence of a single vortex is

$$G_V = \frac{H_c^2 \lambda_L^3}{4\pi} \int d^3r \left\{ -\tfrac{1}{2} + \vec{b}^{\,2} + (\vec{\partial} \times \vec{b})^2 - 2\vec{h} \cdot \vec{b} \right\}$$

$$\frac{G_V - G_0}{L} = \frac{H_c \lambda_L^2}{4\pi} \int d^2\rho \left\{ \vec{b} \cdot \left(\vec{b} + \vec{\partial} \times (\vec{\partial} \times \vec{b})\right) - 2\vec{h} \cdot \vec{b} \right\}$$

The total flux is $\int d^2\rho\, b(\vec{\rho}) = -2\pi n \kappa^{-1}$, which we can see by integrating after taking the curl

$$\vec{\partial}\times(\vec{\partial}\times\vec{b}) = -\vec{\partial}\times(K^{-1}\vec{\partial}\varphi+\vec{a})$$

$$\Rightarrow \quad \vec{b}+\vec{\partial}\times(\vec{\partial}\times\vec{b}) = -2\pi K^{-1} n\,\delta(\vec{r})\,\hat{z}$$

Recall we had found $b(\vec{\rho}) = -nK^{-1}K_0(\rho)$ for a vortex of strength n. We replace

$$\vec{b}\cdot\{\vec{b}+\vec{\partial}\times(\vec{\partial}\times\vec{b})\} = -2\pi K^{-1} n\,\delta(\vec{r})\,b(\vec{r})$$

$$\rightarrow \; -2\pi K^{-1} n\,\delta(\vec{r})\,b(\xi)$$

and we replace $b(0)\rightarrow b(\xi/\lambda_L) = b(K^{-1})$ to get

$$\frac{G_v-G_0}{L} = \frac{H_c^2\lambda_L^2}{4\pi}\left\{\frac{2\pi n^2}{K^2}\ln\big(2\overbrace{e^{-C}}^{1.123}K\big)+\frac{4\pi n h}{K}\right\}$$

in the limit $K\gg1$ (extreme type II). This expression is positive definite for $h=0$, and for $n=-1$ the single vortex energy per unit length goes below that of the bulk superconductor whe

$$h = h_{c1} \equiv \frac{1}{2}K^{-1}\ln(2e^{-C}K)$$

Here $C=0.511\ldots$ we have $e^{-C}\approx 1.123$, and restoring units,

$$H_{c1} = \frac{H_c}{\sqrt{2}\,K}\ln(2e^{-C}K) = \frac{\phi_L}{4\pi\lambda_L^2}\ln(1.123K)$$

and for $K\gg1$, with $H_c = \sqrt{2}\,K\,\phi_L/4\pi\lambda_L^2$, we obtain

$$H_{c1} = \frac{\ln(1.123K)}{\sqrt{2}\,K}H_c \quad , \quad H_{c2} = \sqrt{2}\,K\,H_c$$

Thus, if E_v is the energy of a single vortex, the lower critical field H_{c1} is given by the relation $H_{c1} = 4\pi E_v/\phi_L$.

Abrikosov vortex lattice: Consider again the linearized Ginzburg–Landau equation,

$$-(\kappa^{-1}\vec{\partial}+i\vec{a})^2\psi = \psi$$

with $\vec{b} = \vec{\partial}\times\vec{a} = b\hat{z}$ and $b=\kappa$ (i.e. $B = \sqrt{2}\,H_c\,\kappa = H_{c2}$).
Choose the symmetric gauge $\vec{a} = -\frac{1}{2}by\,\hat{x} + \frac{1}{2}bx\,\hat{y}$. Recall

$$\hat{L} = -(\kappa^{-1}\vec{\partial}+i\vec{a})^2 = \frac{2b}{\kappa}\left(\gamma^\dagger\gamma+\frac{1}{2}\right) - \frac{1}{\kappa^2}\frac{\partial^2}{\partial z^2}$$

with

$$\gamma = \frac{\sqrt{2}}{i\kappa}\left(\frac{\partial}{\partial w}+\frac{1}{4}\kappa^2\bar{w}\right) = \sqrt{\frac{\kappa}{2b}}\,(\pi_x - i\pi_y)$$

where $w \equiv x+iy$ and $\bar{w} \equiv x-iy$ are complex coordinates.
We see that

$$\gamma = \frac{\sqrt{2}}{i\kappa}\,e^{-\kappa^2\bar{w}w/4}\,\frac{\partial}{\partial w}\,e^{+\kappa^2\bar{w}w/4}$$

Thus we conclude any function $\psi(x,y)$ satisfying $\gamma\psi(x,y)=0$ must be of the form

$$\psi(x,y) = f(\bar{w})\,e^{-\kappa w\bar{w}/4}$$

which is to say it is a product of a Gaussian $e^{-\kappa\bar{w}w/4}$ and a function $f(\bar{w})$ which is analytic in $\bar{w}$ (i.e. the function f is **antiholomorphic**. Note $\bar{w}w = x^2+y^2 = \rho^2$.
We define $\psi_0(\vec{\rho}) = \left(\frac{\kappa}{4\pi}\right)^{1/2} e^{-\kappa(x^2+y^2)/4}$, the ground state of $\hat{L}$.

Thus,

$$\psi_{n,k_z}(x,y,z) = \frac{1}{\sqrt{L_z}} e^{ik_z z} \frac{(\gamma^\dagger)^n}{\sqrt{n!}} \psi_0(\vec{\rho})$$

is a normalized excited state of $\hat{L}$ with eigenvalue

$$\varepsilon_n(k_z) = \frac{k_z^2}{K^2} + (2n+1)\frac{b}{K}$$

The ground state has $n=0$, $k_z=0$. However, each such **Landau level** is massively degenerate! We have thus far missed another pair of ladder operators: $\partial_w^\dagger = -\partial_{\bar{w}}$

$$\gamma = \frac{\sqrt{2}}{iK}\left(\frac{\partial}{\partial w} + \frac{1}{4}K^2\bar{w}\right), \quad \gamma^\dagger = \frac{\sqrt{2}}{iK}\left(\frac{\partial}{\partial \bar{w}} - \frac{1}{4}K^2 w\right)$$

$$\beta = \frac{\sqrt{2}}{iK}\left(\frac{\partial}{\partial \bar{w}} + \frac{1}{4}K^2 w\right), \quad \beta^\dagger = \frac{\sqrt{2}}{iK}\left(\frac{\partial}{\partial w} - \frac{1}{4}K^2\bar{w}\right)$$

You can check $[\gamma,\gamma^\dagger] = [\beta,\beta^\dagger] = 1$ but $[\gamma,\beta] = [\gamma,\beta^\dagger] = 0$, and that $\beta\psi_0(\vec{r}) = 0$. Thus, the full set of eigenstates of $\hat{L}$ is given by

$$\psi_{n,m,k_z}(\vec{r}) = \frac{1}{\sqrt{L_z}} e^{ik_z z} \frac{(\gamma^\dagger)^n(\beta^\dagger)^m}{\sqrt{n!\,m!}} \psi_0(\vec{r})$$

with eigenvalues

$$\varepsilon_{n,m}(k_z) = \frac{k_z^2}{K^2} + (2n+1)\frac{b}{K}$$

independent of the index m. The freedom to choose <u>any</u> antiholomorphic function $f(\bar{w})$ as a representative of the lowest Landau level is associated with this degeneracy. In particular, any function of the form

$$f(\bar{w}) = C \prod_{i=1}^{N_v} (\bar{w} - \bar{w}_i)$$

satisfies $\gamma f = 0$. The constants $\{\bar{w}_i\}$ are the complexified locations of N_v **antivortices**. Note that

$$|\psi(x,y)|^2 = |C|^2 e^{-\kappa \bar{w} w/2} \prod_{i=1}^{N_v} |\bar{w} - \bar{w}_i|^2 \equiv |C|^2 e^{-\Phi(\vec{\rho})}$$

where

$$\Phi(\vec{\rho}) = \tfrac{1}{2} \kappa^2 \vec{\rho}^{\,2} - 2 \sum_i \ln |\vec{\rho} - \vec{\rho}_i|$$

Thus, $\Phi(\vec{\rho})$ may be interpreted as the electrostatic potential of a group of N_v point charges in two dimensions, in the presence of a uniform background $\left(\vec{\partial}^2 \frac{1}{2} \kappa \vec{\rho}^{\,2} = 2\kappa\right)$. In the thermodynamic limit, we demand $|\psi|^2$ have a constant density (averaged locally), so

$$\nabla^2 \Phi(\vec{\rho}) = 2\kappa^2 - 4\pi \sum_{i=1}^{N_v} \delta(\vec{\rho} - \vec{\rho}_i) \qquad (\equiv 0 \text{ on average})$$

and

$$\langle n(\vec{\rho}) \rangle = \left\langle \sum_{i=1}^{N_v} \delta(\vec{\rho} - \vec{\rho}_i) \right\rangle = \frac{\kappa^2}{2\pi} \qquad n(\vec{\rho}) = \sum_{i=1}^{N_v} \delta(\vec{\rho} - \vec{\rho}_v)$$

Each antivortex carries one London flux quantum in physical units. In our dimensionless units, the flux is $2\pi/\kappa$ per (anti)vortex, since $\int d^2\rho \, b(\vec{\rho}) = -\frac{2\pi}{\kappa} n \hat{z}$.

Just below the upper critical field we may write

$$\psi = \psi_0 + \delta\psi \quad , \quad b = \kappa + \delta b \quad , \quad \delta b = h - \kappa - \frac{|\psi_0|^2}{2\kappa}$$

Here, $\delta b < 0$. The last equation comes from the second Ginzburg–Landau eqn, with $\vec{\pi} \equiv -i\kappa^{-1}\vec{\partial} + \vec{a}$,

$$\vec{\partial} \times (\vec{h} - \vec{b}) = \frac{1}{2}\left(\psi^* \vec{\pi}\, \psi + \psi \vec{\pi}^* \psi^*\right) = \mathrm{Re}\left(\psi^* \vec{\pi}\, \psi\right)$$

At this point the solution becomes a bit detailed, and you can consult Eqns. 11.175 – 186 in the lecture notes for details. We arrive at an equation,

$$\int d^2r \left\{ \left(\frac{h}{\kappa} - 1\right) |\psi_0(\vec{r})|^2 + \left(1 - \frac{1}{2\kappa^2}\right) |\psi_0(\vec{r})|^4 \right\} = 0$$

This tells us how we must arrange the zeroes (antivortices) in $f(\bar{w})$. Note that it says

$$\left(1 - \frac{h}{\kappa}\right) \langle |\psi_0|^2 \rangle = \left(1 - \frac{1}{2\kappa^2}\right) \langle |\psi_0|^4 \rangle$$

where $\langle \cdots \rangle$ is a spatial average. Now define the ratio

$$\beta_A \equiv \frac{\langle |\psi_0|^4 \rangle}{\langle |\psi_0|^2 \rangle^2} \quad = \text{Abrikosov parameter}$$

Then we have

$$\langle |\psi_0|^2 \rangle = \frac{1}{\beta_A} \frac{\langle |\psi_0|^4 \rangle}{\langle |\psi_0|^2 \rangle} = \frac{2\kappa(\kappa - h)}{(2\kappa^2 - 1)\beta_A}$$

Now let's compute the Gibbs free energy density:

$$g_s - g_n = -\frac{1}{2}\langle|\psi_0|^4\rangle + \langle(b-h)^2\rangle$$

$$= -\frac{1}{2}\left(1-\frac{1}{2\kappa^2}\right)\langle|\psi_0|^4\rangle = -\frac{1}{2}\left(1-\frac{h}{\kappa}\right)\langle|\psi_0|^2\rangle$$

$$= -\frac{(\kappa-h)^2}{2\kappa^2-1}\,\frac{1}{\beta_A}$$

Restore physical units,

$$g_s = -\frac{1}{8\pi}\left\{H^2 + \frac{(H_{c2}-H)^2}{(2\kappa^2-1)\beta_A}\right\}$$

Find $\beta_A^{SQ} = 1.18$, $\beta_A^{TRI} = 1.16$. Avg magnetic field is

$$\langle B\rangle = -4\pi\frac{\partial g_s}{\partial H} = H - \frac{H_{c2}-H}{(2\kappa^2-1)\beta_A}$$

$$\Rightarrow \quad M = \frac{B-H}{4\pi} = \frac{H-H_{c2}}{4\pi(2\kappa^2-1)\beta_A} \quad \Rightarrow$$

$$\chi = \frac{\partial M}{\partial H} = \frac{1}{4\pi\beta_A}\cdot\frac{1}{2\kappa^2-1}$$

Just above $H=H_{c1}$: assume vortex lattice

$$\frac{G_{VL}-G_N}{L} = \frac{H_c\lambda_L^2}{4\pi}\int d^2\rho\,\left\{\vec{b}\cdot\left(\vec{b}+\vec{\partial}\times(\vec{\partial}\times\vec{b})\right) - 2\vec{h}\cdot\vec{b}\right\}$$

We have

$$\vec{b}+\vec{\partial}\times(\vec{\partial}\times\vec{b}) = -\frac{2\pi}{\kappa}\sum_{i=1}^{N_v} n_i\,\delta(\vec{\rho}-\vec{\rho}_i)$$

and

$$\vec{b}(\vec{\rho}) = -\frac{1}{\kappa}\sum_{i=1}^{N_v} n_i K_0\left(\frac{|\vec{\rho}-\vec{\rho}_i|}{\lambda_L}\right)$$

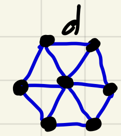
Replace $K_0(0)$ by $K_0(\kappa^{-1})$ to get

$$\frac{G_{VL}-G_N}{L} = \frac{H_c^2\lambda_L^2}{\kappa^2}\left\{\underbrace{\frac{1}{2}\ln(1.123\kappa)\sum_{i=1}^{N_v} n_i^2}_{\text{self-interaction}} + \overbrace{\sum_{i<j}^{N_v} n_i n_j K_0\left(\frac{|\vec{\rho}_i-\vec{\rho}_j|}{\lambda_L}\right)}^{\text{v-v interaction}} + \underbrace{\kappa h\sum_i n_i}_{\substack{\text{external field} \\ -\vec{B}\cdot\vec{H}/4\pi}}\right.$$

If $H-H_{c1} \ll H_{c1}$, vortices are spread far apart, can consider only nearest neighbor vortex pairs in the interaction term. Assume $n_i = -1$ for all vortices. Then

$$\frac{G_{VL}-G_N}{L} = \frac{N_v H_c^2\lambda_L^2}{\kappa^2}\left\{\frac{1}{2}\ln(1.123\kappa) + \frac{1}{2}\overset{\text{\# of NN}}{z}K_0(\overset{\text{NN separation}}{d}) - \kappa h\right\}$$

What is d? Write $N_v = A/\overset{\text{unit cell area of vortex lattice}}{\Omega}$



E.g. triangular lattice $\Rightarrow$ $z = 6$, $\Omega = \frac{\sqrt{3}}{2}d^2$, so

$$\frac{G_{VL}-G_0}{L} = \frac{H_c^2\lambda_L^2}{\sqrt{3}\,\kappa^2}\left\{\left(\ln(1.123\kappa) - 2\kappa h\right)d^{-2} + 6d^{-2}K_0(d)\right\}$$