$$\hat{M} = \frac{1}{2}\hat{N} = \#\text{ Cooper pairs} \Rightarrow \hat{M} \leftrightarrow \frac{1}{i}\frac{\partial}{\partial\alpha}$$

Project onto state of definite particle number $N = 2M$:

$$|M\rangle = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi}\, e^{-iM\alpha}\, |\alpha\rangle$$

Number fluctuations:

$$\frac{\langle\alpha|\hat{N}^2|\alpha\rangle - \langle\alpha|\hat{N}|\alpha\rangle^2}{\langle\alpha|\hat{N}|\alpha\rangle} = \frac{2\int d^3k\, \sin^2\vartheta_{\mathbf{k}} \cos^2\vartheta_{\mathbf{k}}}{\int d^3k\, \sin^2\vartheta_{\mathbf{k}}}$$

Thus, $\Delta N_{RMS} \propto \sqrt{\langle N\rangle}$

Lecture 18 (March 2)

Finite temperature: The finite temperature gap equation is

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}} \tanh\left(\frac{E_{\mathbf{k}'}}{2k_B T}\right)$$

As $T \to \infty$, we see that $\Delta_{\mathbf{k}} = 0$ is the only solution. At what temperature does the gap collapse? We again take

$$V_{\mathbf{k},\mathbf{k}'} = -\frac{v}{V}\,\Theta(\hbar\omega_D - |\xi_{\mathbf{k}}|)\,\Theta(\hbar\omega_D - |\xi_{\mathbf{k}'}|)$$

$$\Delta_{\mathbf{k}} = \Delta\,\Theta(\hbar\omega_D - |\xi_{\mathbf{k}}|)$$

and we obtain

$$1 = \frac{1}{2} g(\varepsilon_F)\, v \int_0^{\hbar\omega_D} d\xi\, (\xi^2 + \Delta^2)^{-1/2} \tanh\left(\frac{(\xi^2+\Delta^2)^{1/2}}{k_B T}\right)$$

Setting $\Delta(T_c) = 0$, we have

$$\int_0^{\hbar\omega_D/2k_BT_c} ds\, s^{-1} \tanh s = \frac{2}{g(\varepsilon_F)\,v}$$

This is an implicit equation for T_c. Assuming $k_BT_c \ll \hbar\omega_D$,

$$\int_0^{\hbar\omega_D/2k_BT_c} ds\, s^{-1} \tanh s = \ln\left(\frac{2e^C}{\pi}\cdot\frac{\hbar\omega_D}{k_BT}\right) + \mathcal{O}\left(e^{-\hbar\omega_D/2k_BT}\right)$$

with $C = 0.57721566\ldots$ the Euler-Mascheroni constant. So

$$k_BT_c = \underbrace{\frac{2e^C}{\pi}}_{1.134}\hbar\omega_D\, e^{-2/g(\varepsilon_F)\,v}$$

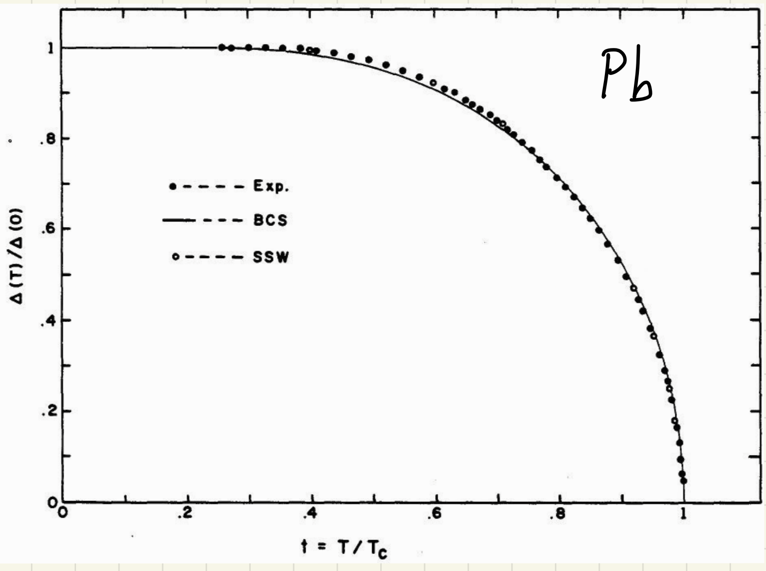
Recall how in this limit we also have

$$\Delta(T=0) = 2\hbar\omega_D\, e^{-2/g(\varepsilon_F)\,v}$$

so combining these results we find the famous relation

$$2\Delta(0) = 2\pi e^{-C} k_BT_c = 3.52\, k_BT_c$$

which relates the $T=0$ gap in the electronic spectrum to T_c.



Isotope effect: The logarithm of the T_c equation is

$$\ln T_c = \ln \omega_D - \frac{2}{g(\varepsilon_F)\,v} + \text{const.}$$

Suppose we vary the mass of the ions via isotopic substitution. Then $\omega_D \propto M^{-1/2}$ changes, but $g(\varepsilon_F)$ and v are largely insensitive. Then

$$\delta \ln T_c = \delta \ln \omega_D = -\tfrac{1}{2}\,\delta \ln M$$

where M is the ionic mass. This relation is fairly well-established among low-T_c materials.

Landau free energy of a superconductor: Let's derive an expression for the free energy of the superconductor as an expansion in the gap Δ. We start with

$$\hat{K}_{BCS} = \sum_{\mathbf{k},\sigma} E_{\mathbf{k}}\, \gamma^\dagger_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - E_{\mathbf{k}}\right) - \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \rangle \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle$$

and the relation

$$\langle c_{-\mathbf{k}-\sigma}\, c_{\mathbf{k}\sigma} \rangle = -\frac{\sigma \Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} \tanh\left(\tfrac{1}{2}\beta E_{\mathbf{k}}\right)$$

which we derived previously. After invoking the gap equation,

$$\hat{K}_{BCS} = \sum_{\mathbf{k},\sigma} E_{\mathbf{k},\sigma}\, \gamma^\dagger_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left\{ \xi_{\mathbf{k}} - E_{\mathbf{k}} + \frac{|\Delta_{\mathbf{k}}|^2}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2k_BT}\right) \right\}$$

Now compute $\Omega_s = -k_BT \ln \mathrm{Tr}\, \exp(-\hat{K}_{BCS}/k_BT)$. We find

$$\Omega_s = -2k_BT \sum_{\mathbf{k}} \ln\left(1 + e^{-E_{\mathbf{k}}/k_BT}\right) + \sum_{\mathbf{k}} \left\{ \xi_{\mathbf{k}} - E_{\mathbf{k}} + \frac{|\Delta_{\mathbf{k}}|^2}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2k_BT}\right) \right\}$$

The normal state free energy is obtained by setting $\Delta_{\mathbf{k}} \to 0$:

$$\Omega_n = -2k_BT \sum_{\mathbf{k}} \ln\left(1 + e^{-|\xi_{\mathbf{k}}|/k_BT}\right) + \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - |\xi_{\mathbf{k}}|\right)$$

With $\Delta_{\mathbf{k}}(T) = \Delta(T)\,\Theta(\hbar\omega_D - |\xi_{\mathbf{k}}|)$, and assuming $\Delta(T) \ll \hbar\omega_D$, we obtain

$$\frac{\Omega_s - \Omega_n}{V} = -\frac{1}{4} g(\varepsilon_F)\Delta^2 \left\{ 1 + 2\ln\left(\frac{\Delta_0}{\Delta}\right) - \left(\frac{\Delta}{2\hbar\omega_D}\right)^2 + \mathcal{O}(\Delta^4) \right\}$$
$$- 2g(\varepsilon_F)\, k_BT\, \Delta \int_0^\infty ds\, \ln\left(1 + e^{-\sqrt{1+s^2}\,\Delta/k_BT}\right) + \frac{\pi^2}{6} g(\varepsilon_F)(k_BT)^2 + \ldots$$

where $\Delta_0 = \Delta(T=0) = \pi e^{-C} k_B T_c$.

① The limit $T \to 0^+$: We have

$$\frac{\Omega_s - \Omega_n}{V} = -\frac{1}{4} g(\varepsilon_F)\Delta^2 \left\{ 1 + 2\ln\left(\frac{\Delta_0}{\Delta}\right) + \mathcal{O}(\Delta^2) \right\}$$
$$- g(\varepsilon_F)\sqrt{2\pi (k_BT)^3 \Delta}\; e^{-\Delta/k_BT} + \frac{\pi^2}{6} g(\varepsilon_F)(k_BT)^2 + \ldots$$

Extremize wrt Δ to obtain an equation for $\Delta(T)$ at low temperatures:

$$\Delta(T) = \Delta_0 - \sqrt{2\pi k_BT\Delta_0}\; e^{-\Delta_0/k_BT} + \ldots$$

Substituting this result into the formula for the free energy difference, we find

$$\frac{\Omega_s - \Omega_n}{V} = -\frac{1}{4} g(\varepsilon_F)\Delta_0^2 + \frac{\pi^2}{6} g(\varepsilon_F)(k_BT)^2$$
$$- g(\varepsilon_F)\sqrt{2\pi (k_BT)^3 \Delta_0}\; e^{-\Delta_0/k_BT} + \ldots$$

With $\Delta_0 = \pi e^{-C} k_B T_c$, setting the above to the condensation energy $-H_c^2(T)/8\pi$ gives

$$H_c(T) = H_c(0)\left\{1 - \underbrace{\tfrac{1}{3}e^{2C}}_{1.057}\left(\frac{T}{T_c}\right)^2 + \ldots\right\}$$

where $H_c(0) = \sqrt{2\pi g(\varepsilon_F)}\,\Delta_0$.

② The limit $T \to T_c^-$: The analysis here is tricky; see §12.14 of the lecture notes. One finds $\zeta(3) = 1.20205690\ldots$

$$\frac{\Omega_S - \Omega_n}{V} = \frac{1}{2} g(\varepsilon_F) \ln\left(\frac{T}{T_c}\right)\Delta^2 + \frac{7\zeta(3)}{32\pi^2}\frac{g(\varepsilon_F)}{(k_B T)^2}\Delta^4 + \mathcal{O}(\Delta^6)$$

$$\equiv \tilde{a}(T)\,\Delta^2 + \frac{1}{2}\tilde{b}(T)\,\Delta^4 + \mathcal{O}(\Delta^6)$$

with, working to lowest order in $T - T_c$,

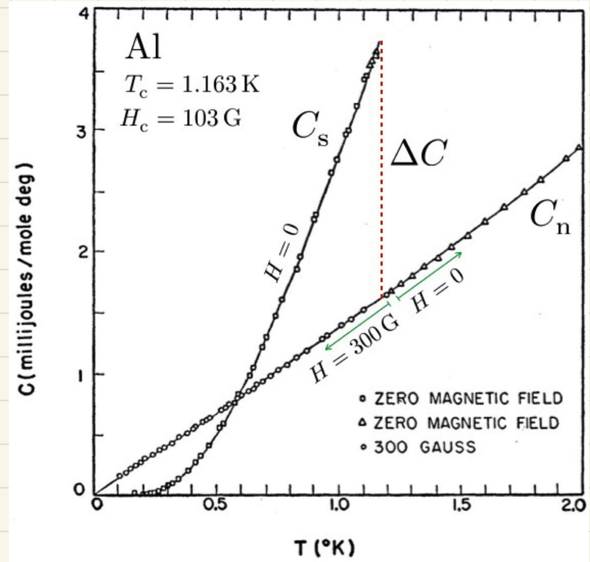
$$\tilde{a}(T) = \frac{1}{2} g(\varepsilon_F)\left(\frac{T}{T_c} - 1\right) \quad , \quad \tilde{b}(T_c) = \frac{7\zeta(3)}{32\pi^2}\frac{g(\varepsilon_F)}{(k_B T_c)^2}$$

The heat capacity jump is then

$$C_s(T_c^-) - C_n(T_c^+) = \frac{T_c\,[\tilde{a}'(T_c)]^2}{\tilde{b}(T_c)} = \frac{4\pi^2}{7\zeta(3)}\,g(\varepsilon_F)\,k_B^2 T_c$$

We then have

$$\frac{C_s(T_c^-) - C_n(T_c^+)}{C_n(T_c^+)} = \frac{12}{7\zeta(3)} = 1.43$$



The order parameter is accordingly given by

$$\Delta^2(T) = -\frac{\tilde{a}(T)}{\tilde{b}(T)} = \frac{8e^{2C}}{7\zeta(3)}\left(1-\frac{T}{T_c}\right)\Delta_0^2$$

in which case

$$\frac{\Delta(T)}{\Delta(0)} = \underbrace{\left(\frac{8e^{2C}}{7\zeta(3)}\right)^{1/2}}_{1.734}\left(1-\frac{T}{T_c}\right)^{1/2}$$

Just below T_c, the thermodynamic critical field H_c is given by the expression $H_c^2 = 4\pi\,\tilde{a}^2(T)/\tilde{b}(T_c)$, hence

$$\frac{H_c(T)}{H_c(0)} = 1.734\left(1-\frac{T}{T_c}\right)$$

Paramagnetic susceptibility : Add a weak magnetic field :

$$\hat{H}_1 = -\mu_B H\sum_{k,\sigma} \sigma\, c^\dagger_{k\sigma} c_{k\sigma} = -\mu_B H\sum_{k,\sigma}\sigma\,\gamma^\dagger_{k\sigma}\gamma_{k\sigma}$$

We compute the free energy shift,

$$\Delta\Omega_s(T,V,\mu,H) \equiv \Omega_s(T,V,\mu,H) - \Omega_s(T,V,\mu,0)$$

$$= -k_BT\sum_{k,\sigma}\ln\left(\frac{1+e^{-(E_k+\sigma\mu_BH)/k_BT}}{1+e^{-E_k/k_BT}}\right)$$

$$= -\frac{(\mu_BH)^2}{k_BT}\sum_{k,\sigma}\frac{e^{E_k/k_BT}}{\left(e^{E_k/k_BT}+1\right)^2} + \mathcal{O}(H^4)$$

The magnetic susceptibility is then

Yoshida function

$$\chi_s = -\frac{1}{V}\frac{\partial^2 \Delta\Omega_s}{\partial H^2}\bigg|_{H=0} = g(\varepsilon_F)\mu_B^2\, Y(T)$$

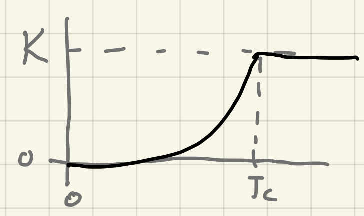
$$Y(T) = 2\int_0^\infty d\xi \left(-\frac{\partial f}{\partial E}\right) = \frac{1}{2k_BT}\int_0^\infty d\xi\, \mathrm{sech}^2\left(\frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}\right)$$

Limits: $Y(T\to 0) \simeq (2\pi\Delta/k_BT)\, e^{-\Delta/k_BT}$ and $Y(T_c) = 1$.

Since $\chi_n(T) = g(\varepsilon_F)\mu_B^2$ = Pauli susceptibility,

$$\frac{\chi_s(T)}{\chi_n(T)} = Y(T)$$

As $T\to 0$, $Y(T)$ is exponentially suppressed as $e^{-\Delta_0/k_BT}$. The susceptibility vanishes exponentially because it requires a finite energy Δ_0 to create a Bogoliubov quasiparticle out of the spin singlet BCS ground state. In metals, nuclear spins experience a shift in their resonance energies due to hyperfine coupling to the conduction electrons, called the **Knight shift**. The formula for the Knight shift is



$$K = \frac{\omega - \omega_0}{\omega_0} = \frac{8\pi}{3}|\psi(0)|^2 \chi_{el}$$

where $\psi(0)$ is the amplitude of the electron wavefunction at the nucleus. In a superconductor, $K(T\to 0)\to 0$ exponentially. Electrons remain unpolarized in a weak external field due to the finite energy gap.

Supercurrent: As we saw within Ginzburg-Landau theory, a spatially varying order parameter $\Psi(\vec{x}) = \Psi_0 \, e^{i\vec{q}\cdot\vec{x}}$, in the absence of external fields, corresponds to finite current density. The free energy density is

$K|\vec{\nabla}\Psi|^2$

$$f = a|\Psi_0|^2 + \tfrac{1}{2}b|\Psi_0|^4 + K\vec{q}^{\,2}|\Psi_0|^2$$

Extremizing wrt $|\Psi_0|^2$ gives $|\Psi_0|^2 = -(a + K\vec{q}^{\,2})/b$ if $a + K\vec{q}^{\,2} < 0$ and zero otherwise. If $a(T) = \alpha(T - T_c^0)$, we conclude $T_c(\vec{q}) = T_c(\vec{q}=0) - \alpha^{-1}K\vec{q}^{\,2}$. The current density is

$$\vec{J} = -\frac{2Ke^*}{\hbar^2}\cdot\frac{\hbar}{2i}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) = -\frac{2Ke^*}{\hbar}\vec{q}$$

To describe a moving condensate within BCS theory, we must give finite momentum to the Cooper pairs. This means

$$\langle c_{-\mathbf{k}+\frac{1}{2}\vec{p}\downarrow}\, c_{\mathbf{k}+\frac{1}{2}\vec{p}\uparrow}\rangle = \Psi_{\mathbf{k},\vec{q}}\,\delta_{\vec{p},\vec{q}}$$

Then

$$\hat{K}_{BCS} = \sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}+\frac{1}{2}\vec{q}\uparrow} & c_{-\mathbf{k}+\frac{1}{2}\vec{q}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{k}+\frac{1}{2}\vec{q}} & \Delta_{\mathbf{k},\vec{q}} \\ \Delta^*_{\mathbf{k},\vec{q}} & \xi_{-\mathbf{k}+\frac{1}{2}\vec{q}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}+\frac{1}{2}\vec{q}\uparrow} \\ c^\dagger_{-\mathbf{k}+\frac{1}{2}\vec{q}\downarrow} \end{pmatrix}$$

$$+ \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - \Delta_{\mathbf{k},\vec{q}}\langle c^\dagger_{\mathbf{k}+\frac{1}{2}\vec{q}\uparrow}\, c^\dagger_{-\mathbf{k}+\frac{1}{2}\vec{q}\downarrow}\rangle\right)$$

The technical details are discussed in §12.11 of the lecture notes. One finds

$$\Delta_{\mathbf{k},\vec{q}} = \Delta_{0,\vec{q}}\,\Theta(\hbar\omega_0 - |\xi_{\mathbf{k}}|)$$

The gap equation is then

$$\sinh^{-1}\left(\frac{\hbar\omega_D+\eta_{\vec{q}}}{\Delta_{0,\vec{q}}}\right)=\frac{2}{g(\varepsilon_F)v}+\sinh^{-1}\left(\frac{\eta_{\vec{q}}}{\Delta_{0,\vec{q}}}\right)$$

with $\eta_{\vec{q}}\equiv\hbar^2\vec{q}^{\,2}/8m^*$ in the case $\varepsilon_{\vec{q}}=\hbar^2\vec{q}^{\,2}/2m^*$. We determine the critical wavevector q_c where the gap collapses by taking $\Delta_{0,\vec{q}}\to 0$, resulting in

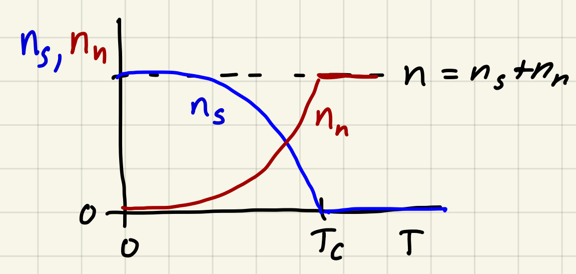
$$\frac{2}{g(\varepsilon_F)v}=\ln\left(1+\frac{\hbar\omega_D}{\eta_{q_c}}\right)\Rightarrow\eta_{q_c}\simeq\hbar\omega_D e^{-2/g(\varepsilon_F)v}=\tfrac{1}{2}\Delta_0$$

Assuming $\eta_{\vec{q}}\ll\Delta_0$, we have the gap $\Delta_{0,\vec{q}}\simeq\Delta_0-\frac{\hbar^2\vec{q}^{\,2}}{8m^*}$.
Now for the super-current. We have

$$\vec{j}=\frac{ne\hbar}{2m^*}\vec{q}+\frac{2e\hbar}{m^*V}\sum_{\vec{k}}\vec{k}\,\langle c^\dagger_{\vec{k}+\frac{1}{2}\vec{q}\uparrow}c_{\vec{k}+\frac{1}{2}\vec{q}\uparrow}\rangle$$

where $n=N/V$. One obtains $\vec{j}=n_s(T)\,e\hbar\vec{q}/2m^*$ where the **superfluid density** $n_s(T)$ is given by

$$n_s(T)=n\left\{1+2\int_0^\infty\! d\xi\,\frac{\partial f}{\partial E}\right\}$$
$$=n\left\{1-Y(T)\right\}$$



where $Y(T)$ is the Yoshida function. Note then $n_n(T)=n\,Y(T)$. Recall $Y(T_c)=1$. Finally, the GL free energy density is

$$\frac{\Omega_s-\Omega_n}{V}=\tilde{a}(T)|\Delta|^2+\tfrac{1}{2}\tilde{b}(T_c)|\Delta|^4+\tilde{K}|\Delta|^2\vec{q}^{\,2}$$

with

$$\tilde{K}=\frac{\hbar^2}{2m^*}\,\frac{n\,\tilde{b}(T_c)}{g(\varepsilon_F)}$$

Effect of repulsive interactions: Let's now take

$$V_{k,k'} = \begin{cases} (v_c - v_p)/V & \text{if } |\xi_k| < \hbar\omega_D \text{ and } |\xi_{k'}| < \hbar\omega_D \\ v_c/V & \text{otherwise} \end{cases}$$

Here v_p is due to phonon-mediated attraction and v_c due to Coulomb repulsion ($v_p, v_c > 0$). We posit a solution

$$\Delta_k = \begin{cases} \Delta_0 & \text{if } |\xi_k| < \hbar\omega_D \\ \Delta_1 & \text{if } |\xi_k| \geqslant \hbar\omega_D \end{cases}$$

with both $\Delta_{0,1} \in \mathbb{R}$. We presume $v_p > v_c > 0$ so that attraction wins close to the Fermi surface, but as we shall see below, this is not absolutely necessary! The gap equation then yields

$$\Delta_0 = \frac{1}{2} g(\varepsilon_F)(v_p - v_c) \int_0^{\hbar\omega_D} d\xi \, \frac{\Delta_0}{\sqrt{\xi^2 + \Delta_0^2}} - \frac{1}{2} g(\varepsilon_F)\, v_c \int_{\hbar\omega_D}^{B} d\xi \, \frac{\Delta_1}{\sqrt{\xi^2 + \Delta_1^2}}$$

$$\Delta_1 = -\frac{1}{2} g(\varepsilon_F)\, v_c \int_0^{\hbar\omega_D} d\xi \, \frac{\Delta_0}{\sqrt{\xi^2 + \Delta_0^2}} - \frac{1}{2} g(\varepsilon_F)\, v_c \int_{\hbar\omega_D}^{B} d\xi \, \frac{\Delta_1}{\sqrt{\xi^2 + \Delta_1^2}}$$

where B is the electron bandwidth. Assuming $|\Delta_{0,1}| \ll \hbar\omega_D \ll B$ we obtain

$$\Delta_0 = \frac{1}{2} g(\varepsilon_F)(v_p - v_c)\, \Delta_0 \ln\left(\frac{2\hbar\omega_D}{\Delta_0}\right) - \frac{1}{2} g(\varepsilon_F)\, v_c\, \Delta_1 \ln\left(\frac{B}{\hbar\omega_D}\right)$$

$$\Delta_1 = -\frac{1}{2} g(\varepsilon_F)\, v_c\, \Delta_0 \ln\left(\frac{2\hbar\omega_D}{\Delta_0}\right) - \frac{1}{2} g(\varepsilon_F)\, v_c\, \Delta_1 \ln\left(\frac{B}{\hbar\omega_D}\right)$$

The second of these equations gives

$$\Delta_1 = -\frac{\frac{1}{2} g(\varepsilon_F) v_c \ln(2\hbar\omega_D/\Delta_0)}{1+\frac{1}{2} g(\varepsilon_F) v_c \ln(B/\hbar\omega_D)} \Delta_0 < 0$$

Insert this into the first equation to find

$$\frac{2}{g(\varepsilon_F) v_p} = \left\{ 1 - \frac{v_c}{v_p} \frac{1}{1+\frac{1}{2} g(\varepsilon_F) v_c \ln(B/\hbar\omega_D)} \right\} \cdot \ln\left(\frac{2\hbar\omega_D}{\Delta_0}\right)$$

which has a solution provided

$$v_p > \frac{v_c}{1+\frac{1}{2} g(\varepsilon_F) v_c \ln(B/\hbar\omega_D)}$$

The RHS reflects a renormalized value of the bare Coulomb repulsion v_c. Thus, we can have a superconducting solution even when

$$v_c > v_p > \frac{v_c}{1+\frac{1}{2} g(\varepsilon_F) v_c \ln(B/\hbar\omega_D)}$$

and the interactions are always repulsive!! Very surprising!

To find T_c, set $\Delta_{0,1} \to 0$ with $r \equiv \Delta_1/\Delta_0$ finite:

$$\frac{2}{g(\varepsilon_F)} = (v_p - v_c) \int_0^{\tilde{\Omega}} ds\, s^{-1} \tanh s - r v_c \int_{\tilde{\Omega}}^{\tilde{B}} ds\, s^{-1} \tanh s$$

$$\frac{2}{g(\varepsilon_F)} = -r^{-1} v_c \int_0^{\tilde{\Omega}} ds\, s^{-1} \tanh s - v_c \int_{\tilde{\Omega}}^{\tilde{B}} ds\, s^{-1} \tanh s$$

where $\tilde{\Omega} \equiv \hbar\omega_D/2k_B T_c$ and $\tilde{B} \equiv B/2k_B T_c$. We obtain

$$\frac{2}{g(\varepsilon_F)v_p} = \left\{1 - \frac{v_c}{v_p}\,\frac{1}{1+\frac{1}{2}g(\varepsilon_F)v_c\ln(B/\hbar\omega_D)}\right\}\cdot\ln\left(\frac{1.134\hbar\omega_D}{k_BT_c}\right)$$

Once again, $2\Delta_0(T=0) = 3.52\,k_BT_c$, but with

$$k_BT_c = 1.134\hbar\omega_D\, e^{-2/g(\varepsilon_F)v_{eff}}$$

and

$$v_{eff} = v_p - \frac{v_c}{1+\frac{1}{2}g(\varepsilon_F)v_c\ln(B/\hbar\omega_D)}$$

Standard notation:

$$\lambda \equiv \frac{1}{2}g(\varepsilon_F)v_p \quad , \quad \mu \equiv \frac{1}{2}g(\varepsilon_F)v_c \quad , \quad \mu^* \equiv \frac{\mu}{1+\mu\ln(B/\hbar\omega_D)}$$

so that

$$k_BT_c = 1.134\hbar\omega_D\, e^{-1/(\lambda-\mu^*)}$$

and

$$\Delta_0 = 2\hbar\omega_D\, e^{-1/(\lambda-\mu^*)} \quad , \quad \Delta_1 = -\frac{\mu^*\Delta_0}{\lambda-\mu^*}$$

Since μ^* depends on ω_D, the isotope effect is affected.