PHYSICS 110A : MECHANICS 1
PROBLEM SET #3 SOLUTIONS

There are four problems in all. Problems 1 and 2 constitute a practice midterm exam.

[**1**] A particle of mass m moves in the one-dimensional potential

$$U(x) = \frac{U_0}{a^4}\left(x^2 - a^2\right)^2 \quad . \tag{1}$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.
`[15 points]`

(b) Sketch a representative set of phase curves. Be sure to sketch any separatrices which exist, and identify their energies. Also sketch all the phase curves for motions with total energy $E = \frac{1}{2}U_0$. Do the same for $E = 2U_0$.
`[15 points]`

(c) The phase space dynamics are written as $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$. Find the upper and lower components of the vector field $\boldsymbol{V}$.
`[10 points]`

(d) Derive an expression for the period T of the motion when the system exhibits small oscillations about a potential minimum.
`[10 points]`

Solution :

(a) Clearly the minima lie at $x = \pm a$ and there is a local maximum at $x = 0$.

(b) See Fig. 1 for the phase curves. Clearly $U(\pm a) = 0$ is the minimum of the potential, and $U(0) = U_0$ is the local maximum and the energy of the separatrix. Thus, $E = \frac{1}{2}U_0$ cuts through the potential in both wells, and the phase curves at this energy form two disjoint sets. For $E < U_0$ there are four turning points, at

$$x_{1,-} = -a\sqrt{1 + \sqrt{\frac{E}{U_0}}} \quad , \quad x_{1,+} = -a\sqrt{1 - \sqrt{\frac{E}{U_0}}}$$

and

$$x_{2,-} = a\sqrt{1 - \sqrt{\frac{E}{U_0}}} \quad , \quad x_{2,+} = a\sqrt{1 + \sqrt{\frac{E}{U_0}}}$$

For $E = 2U_0$, the energy is above that of the separatrix, and there are only two turning points, $x_{1,-}$ and $x_{2,+}$. The phase curve is then connected.

(c) From $m\ddot{x} = -U'(x)$ we have

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{1}{m}U'(x) \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{4U_0}{ma^4}\,x\,(x^2 - a^2) \end{pmatrix} \quad . \tag{2}$$

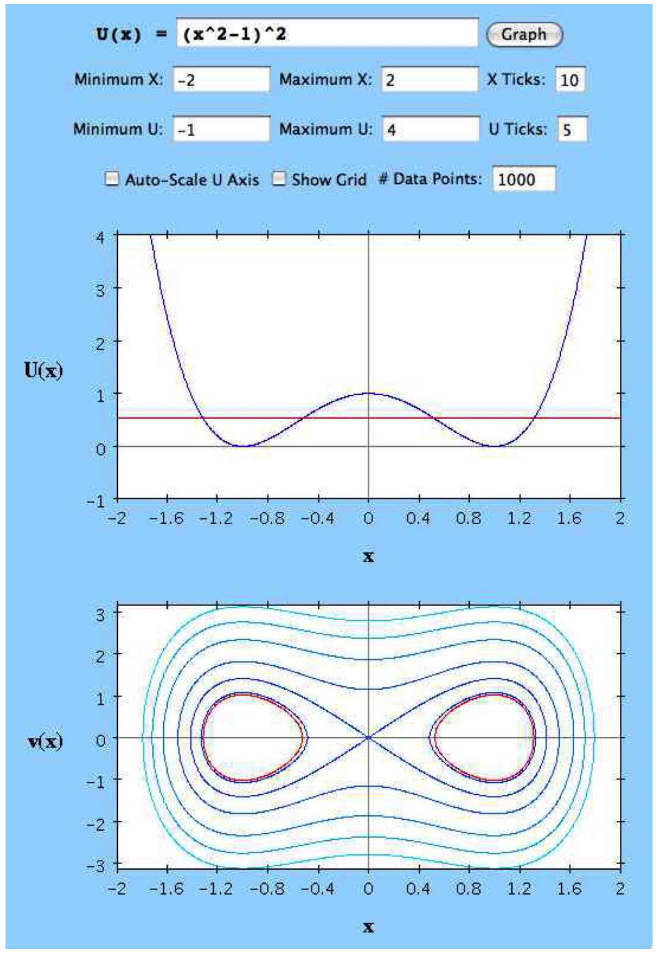

Figure 1: Sketch of the double well potential $U(x) = (U_0/a^4)(x^2 - a^2)^2$, here with distances in units of a, and associated phase curves. The separatrix is the phase curve which runs through the origin. Shown in red is the phase curve for $U = \frac{1}{2}\,U_0$, consisting of two deformed ellipses. For $U = 2\,U_0$, the phase curve is connected, lying outside the separatrix.

(d) Set $x = \pm a + \eta$ and Taylor expand:

$$U(\pm a + \eta) = \frac{4U_0}{a^2}\,\eta^2 + \mathcal{O}(\eta^3) \quad . \tag{3}$$

Equating this with $\frac{1}{2}k\,\eta^2$, we have the effective spring constant $k = 8U_0/a^2$, and the small oscillation frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8\,U_0}{ma^2}} \quad . \tag{4}$$

The period is $2\pi/\omega_0$.

[2] An R-L-C circuit is shown in fig. 2. The resistive element is a light bulb. The inductance is $L = 400\,\mu\text{H}$; the capacitance is $C = 1\,\mu\text{F}$; the resistance is $R = 32\,\Omega$. The voltage $V(t)$ oscillates sinusoidally, with $V(t) = V_0 \cos(\omega t)$, where $V_0 = 4\,\text{V}$. In this problem, you may neglect all transients; we are interested in the late time, steady state operation of this circuit. Recall the relevant MKS units:

$$1\,\Omega = 1\,\text{V}\cdot\text{s}\,/\,\text{C} \quad , \quad 1\,\text{F} = 1\,\text{C}\,/\,\text{V} \quad , \quad 1\,\text{H} = 1\,\text{V}\cdot\text{s}^2/\,\text{C} \quad .$$

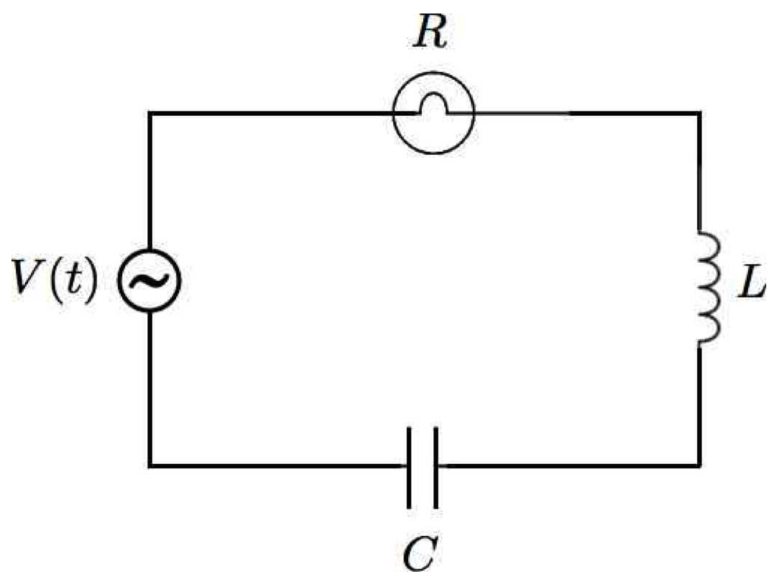

Figure 2: An R-L-C circuit in which the resistive element is a light bulb.

(a) Is this circuit underdamped or overdamped? Why?
`[10 points]`

(b) Suppose the bulb will only emit light when the average power dissipated by the bulb is greater than a threshold $P_{\text{th}} = \frac{2}{9}\,W$. For fixed $V_0 = 4\,\text{V}$, find the frequency range for ω over which the bulb emits light. Recall that the instantaneous power dissipated by a resistor is $P_R(t) = I^2(t)R$. (Average this over a cycle to get the average power dissipated.)
`[20 points]`

(c) Neglecting transients, compare the expressions for the instantaneous power supplied by the voltage source, $P_V(t)$, and the power dissipated by the resistor $P_R(t) = I^2(t)R$. If $P_V(t) \neq P_R(t)$, where does the power extra power go or come from? What can you say about the averages of P_V and $P_R(t)$ over a cycle? Explain your answer.
`[20 points]`

(d) What is the maximum charge Q_{max} on the capacitor plate if $\omega = 3000\,\text{s}^{-1}$?
`[100 quatloos extra credit]`

Solution :

(a) We have

$$\omega_0 = (LC)^{-1/2} = 5 \times 10^4 \, \mathrm{s}^{-1} \quad , \quad \beta = \frac{R}{2L} = 4 \times 10^4 \, \mathrm{s}^{-1} \quad .$$

Thus, $\omega_0^2 > \beta^2$ and the circuit is *underdamped*.

(b) The charge on the capacitor plate obeys the ODE

$$L \ddot{Q} + R \dot{Q} + \frac{Q}{C} = V(t) \quad .$$

The solution is

$$Q(t) = Q_{\mathrm{hom}}(t) + A(\omega) \frac{V_0}{L} \cos\big(\omega t - \delta(\omega)\big) \quad ,$$

with

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\!\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \quad .$$

Thus, ignoring the transients, the power dissipated by the bulb is

$$P_R(t) = \dot{Q}^2(t)\, R = \omega^2 A^2(\omega) \frac{V_0^2 R}{L^2} \sin^2\big(\omega t - \delta(\omega)\big) \quad .$$

Averaging over a period, we have $\langle \sin^2(\omega t - \delta) \rangle = \frac{1}{2}$, so

$$\langle P_R \rangle = \omega^2 A^2(\omega) \frac{V_0^2 R}{2L^2} = \frac{V_0^2}{2R} \cdot \frac{4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \quad .$$

Now $V_0^2/2R = \frac{1}{4}\,\mathrm{W}$. So $P_{\mathrm{th}} = \alpha V_0^2/2R$, with $\alpha = \frac{8}{9}$. We then set $\langle P_R \rangle = P_{\mathrm{th}}$, whence

$$(1-\alpha) \cdot 4\beta^2\omega^2 = \alpha\, (\omega_0^2 - \omega^2)^2 \quad .$$

The solutions are

$$\omega_\pm = \pm\sqrt{\frac{1-\alpha}{\alpha}}\, \beta + \sqrt{\left(\frac{1-\alpha}{\alpha}\right)\beta^2 + \omega_0^2} = \big(3\sqrt{3} \pm \sqrt{2}\big) \times 1000 \, \mathrm{s}^{-1} \quad .$$

So light is emitted for $\omega \in (\omega_-, \omega_+)$.

(c) The instantaneous power supplied by the voltage source is

$$\begin{aligned} P_V(t) = V(t)\, I(t) &= -\omega\, A \frac{V_0^2}{L} \sin(\omega t - \delta) \cos(\omega t) \\ &= \omega\, A \frac{V_0^2}{2L} \Big(\sin\delta - \sin(2\omega t - \delta) \Big) \quad . \end{aligned}$$

As we have seen, the power dissipated by the bulb is

$$P_R(t) = \omega^2 A^2 \frac{V_0^2 R}{L^2} \sin^2(\omega t - \delta) \quad .$$

These two quantities are not identical, but they do have identical time averages over one cycle:

$$\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle = \frac{V_0^2}{2R}\cdot 4\beta^2\,\omega^2\,A^2(\omega) \quad .$$

Energy conservation means

$$P_V(t) = P_R(t) + \dot{E}(t) \quad ,$$

where

$$E(t) = \frac{L\dot{Q}^2}{2} + \frac{Q^2}{2C}$$

is the energy in the inductor and capacitor. Since $Q(t)$ is periodic, the average of $\dot{E}$ over a cycle must vanish, which guarantees $\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle$.

(d) Kirchoff's law gives for this circuit the equation

$$\ddot{Q} + 2\beta\,\dot{Q} + \omega_0^2\,Q = \frac{V_0}{L}\,\cos(\omega t) \quad ,$$

with the solution

$$Q(t) = Q_{\rm hom}(t) + A(\omega)\,\frac{V_0}{L}\,\cos\big(\omega t - \delta(\omega)\big) \quad ,$$

where $Q_{\rm hom}(t)$ is the homogeneous solution, *i.e.* the transient which we ignore, and

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\!\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \quad .$$

Then

$$Q_{\rm max} = A(\omega)\,\frac{V_0}{L} \quad .$$

Plugging in $\omega = 3000\,{\rm s}^{-1}$, we have

$$A(\omega) = \big[(5^2 - 3^2)^2 + 4\cdot 4^2\cdot 3^2\big]^{-1/2}\times 10^{-3}\,{\rm s}^2 = \frac{1}{8\sqrt{13}}\times 10^{-3}\,{\rm s}^2 \quad .$$

Since $V_0/L = 10^4\,{\rm C/s}^2$, we have

$$Q_{\rm max} = \frac{5}{4\sqrt{13}}\,{\rm C} = 0.347\,{\rm C} \quad .$$

[3] The potential energy of two atoms in a molecule can sometimes be approximated by the Morse function,

$$U(r) = A\bigg[\Big(e^{(R-r)/\lambda} - 1\Big)^2 - 1\bigg] \quad ,$$

where r is the interatomic distance and A, R, and λ are positive constants.

(a) Sketch the function $U(r)$ for $0 < r < \infty$.

(b) Find the equilibrium separation r^* at which $U(r)$ is minimized.

(c) Assume the motion is one-dimensional. Writing $r = r^* + x$, so that x is the displacement relative to equilibrium, show that $U(r)$ takes the form $U(r^* + x) = U_0 + \frac{1}{2}kx^2$ for small $|x|$, so that Hooke's law applies. What do we mean by 'small'?

(d) What is the effective force constant k?

Solution :

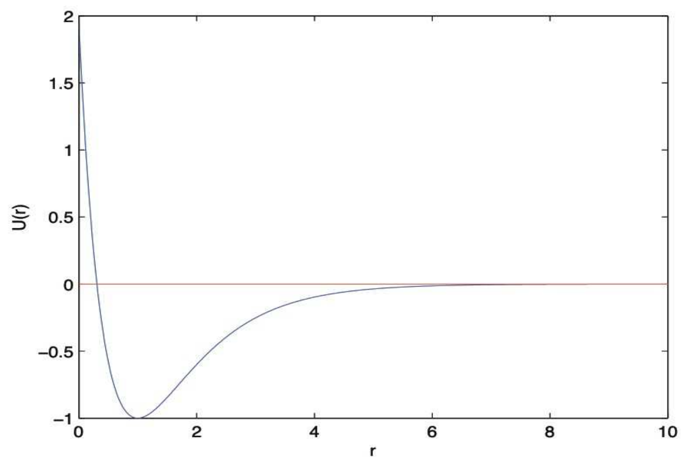

Figure 3: Sketch of Morse potential for $A = R = \lambda = 1$.

(a) See fig. 3.

(b) The equilibrium separation r^* is the solution to the equation $U'(r^*) = 0$. From

$$U'(r) = -\frac{2A}{\lambda}\, e^{(R-r)/\lambda}\Big(e^{(R-r)/\lambda} - 1\Big)$$

we obtain $r^* = R$.

(c) now we expand $U\,(r)$ as a Taylor series about $r = r^* = R$:

$$\begin{aligned} U(R+x) &= A\big(e^{-x/\lambda} - 1\big)^2 - A \\ &= A\Big(-\frac{x}{\lambda} + \frac{x^2}{2\lambda^2} + \dots\Big)^2 - A \\ &= -A + A\,\frac{x^2}{\lambda^2} - A\,\frac{x^3}{\lambda^3} + \mathcal{O}(x^4) \quad , \end{aligned}$$

from which we determine $U_0 = -A$ and $k = U''(R) = 2A/\lambda^2$. By 'small' we mean that the third order term in the Taylor expansion is small in comparison with the second order term, which evidently requires $|x| \ll \lambda$.

(d) We have $k = U''(R) = 2A/\lambda^2$.

[4] An undamped oscillator has a period $T = 1.000\,\mathrm{s}$. Some damping is then introduced, causing the period of the damped oscillations to increase to $T' = 1.001\,\mathrm{s}$.

(a) What is the damping coefficient β?

(b) By what factor will the oscillation amplitude be decreased after ten cycles?

(c) Which effect of the damping would be more noticeable: the change in the period, or the change in the amplitude?

Solution :

(a) We have $\omega_0 = 2\pi/T$ and $T' = 2\pi/\nu = 2\pi/\sqrt{\omega_0^2 - \beta^2}$. Thus,

$$\beta = \omega_0 \sqrt{1 - \left(\frac{T}{T'}\right)^2} = 0.281\,\mathrm{s}^{-1} \quad .$$

(b) The amplitude reduction is

$$\exp(-10\beta T') = 0.060 \quad .$$

(c) The amplitude is exponentially attenuated and after ten cycles is affected much more than the frequency.