PHYSICS 110A : MECHANICS 1
PROBLEM SET #9 SOLUTIONS

[1] At perigee of an elliptical orbit, a satellite fires a rocket in the direction of its motion so as to effectively instantaneously increase its speed by a factor $\lambda > 0$.

(a) For what values of λ does the orbit remain bound?

(b) Assuming the orbit remains bound, find the new values of the semimajor axis length $\tilde{a}$, the eccentricity $\tilde{\varepsilon}$, and the location of periapsis $\tilde{\phi}_0$.

(c) For the unbound case, find the opening angle of the hyperbolic orbit.

Solution :

The location of perigee remains the same, because the momentum $\mu\dot{\boldsymbol{r}}$ is orthogonal to the position vector $\boldsymbol{r}$ both before and after the impulse. Thus,

$$a\,(1-\varepsilon) = \tilde{a}\,|1-\tilde{\varepsilon}| \quad .$$

Since $a\,|1-\varepsilon^2| = \ell^2/\mu k$, we have, after dividing by the previous equation,

$$1+\tilde{\varepsilon} = (1+\varepsilon)\cdot\frac{\tilde{\ell}^2}{\ell^2} \quad ,$$

and since $\boldsymbol{\ell} = \mu\boldsymbol{r}\times\dot{\boldsymbol{r}}$, we have $\tilde{\boldsymbol{\ell}} = \lambda\boldsymbol{\ell}$. This yields

$$1+\tilde{\varepsilon} = \lambda^2(1+\varepsilon) \quad .$$

(a) The threshold for a bound orbit is $\tilde{\varepsilon} = 1$, hence the critical value of λ is

$$\lambda_{\mathrm{c}} = \sqrt{\frac{2}{1+\varepsilon}} \quad .$$

(b) Solving for $\tilde{\varepsilon}$ and $\tilde{a}$, we have

$$\tilde{\varepsilon} = \lambda^2(1+\varepsilon) - 1 \qquad , \qquad \tilde{a} = \frac{(1-\varepsilon)\,a}{2-\lambda^2(1+\varepsilon)} \quad .$$

Since perigee does not change, $\tilde{\phi}_0 = \phi_0 = \pi$.

(c) For a hyperbolic orbit, $\tilde{\varepsilon} > 1$, and the opening angle is $2\cos^{-1}(1/\tilde{\varepsilon})$.

[2] Evil space aliens send a probe into our solar system to observe the earth. The probe orbits the sun with its perihelion at distance $r_{\mathrm{p}} = \frac{3}{4}a_{\oplus}$, where its velocity is $v_{\mathrm{p}} = \frac{4}{3}v_{\oplus}$. (The quantities $a_{\oplus}$ and $v_{\oplus}$ are the orbital radius and velocity of the earth, respectively; you may neglect the eccentricity of earth's orbit.) The probe's orbit is coplanar with that of the earth, and you may neglect the interaction of the probe with all bodies other than the sun.

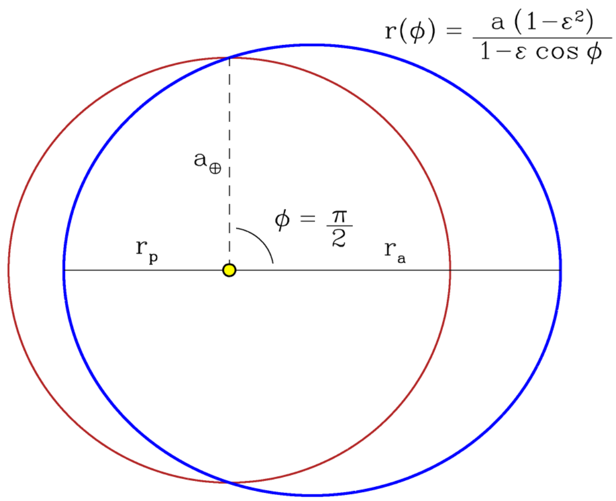

Figure 1: Orbits of the earth (red) and probe (blue).

(a) Compute the eccentricity of the probe's orbit.

(b) Compute the probe's distance from the sun at aphelion, and its velocity at aphelion.

(c) Write down the geometric equation for the probe's orbit.

(d) Let perihelion occur when the azimuthal angle is $\phi = \pi$. At what value of ϕ does the probe cross the earth's orbit?

(e) Compute the period of the probe's orbit.

Solution :

Let's consider the general case where $r_{\rm p} = \alpha\, a_\oplus$ and $v_{\rm p} = \beta\, v_\oplus$. The energy of the probe's orbit is given by

$$E = \tfrac{1}{2}mv_{\rm p}^2 - \frac{GMm}{r_{\rm p}} = \tfrac{1}{2}mv_{\rm a}^2 - \frac{GMm}{r_{\rm a}} \ ,$$

where m is the mass of the probe. The probe's angular momentum is

$$\ell = mr_{\rm p}v_{\rm p} = mr_{\rm a}v_{\rm a} = \alpha\beta\, ma_\oplus v_\oplus \ .$$

It is helpful to express all velocities in terms of $v_\oplus$ and all distances in units of $a_\oplus$. To this end, we write

$$GM = a_\oplus\, v_\oplus^2 \ ,$$

where M is the solar mass. Using the angular momentum equation to eliminate $v_{\rm a}$ in terms of $r_{\rm a}$, we obtain the equation

$$\frac{\alpha^2\beta^2}{2\rho^2} - \frac{1}{\rho} + \frac{1}{\alpha} - \frac{\beta^2}{2} = 0 \ ,$$

where $\rho \equiv r_\mathrm{a}/a_\oplus$. This is a quadratic equation for ρ, the solutions of which are $\rho = \alpha$ (trivial) and

$$\rho = \frac{\alpha^2\beta^2}{2-\alpha\beta^2} \ .$$

If $\rho > \alpha$ then r_a corresponds to aphelion and r_p to perihelion. For $\alpha = \frac{3}{4}$ and $\beta = \frac{4}{3}$, we have $\rho = \frac{3}{2}$.

(a) The eccentricity of the probe's orbit is

$$\varepsilon = \frac{r_\mathrm{a} - r_\mathrm{p}}{r_\mathrm{a} + r_\mathrm{p}} = \alpha\beta^2 - 1 \ .$$

For $\alpha = \frac{3}{4}$ and $\beta = \frac{4}{3}$, we find $\varepsilon = \frac{1}{3}$.

(b) The distance at aphelion is $r_\mathrm{a} = \rho\, a_\oplus = \frac{3}{2}\,\mathrm{AU}$. The probe's velocity at aphelion is then $v_\mathrm{a} = \alpha\beta v_\oplus/\rho = (2-\alpha\beta^2)v_\oplus/\alpha\beta = \frac{2}{3}\,v_\oplus$.

(c) The geometric equation of the probe's orbit is

$$r(\phi) = \frac{a\,(1-\varepsilon^2)}{1-\varepsilon\,\cos\phi} \ ,$$

where

$$a = \tfrac{1}{2}(r_\mathrm{a} + r_\mathrm{p}) = \frac{\alpha\, a_\oplus}{2-\alpha\beta^2} \qquad , \qquad \varepsilon = \frac{r_\mathrm{a} - r_\mathrm{p}}{r_\mathrm{a} + r_\mathrm{p}} = \alpha\beta^2 - 1 \ .$$

(d) The probe intersect's the earth's orbit when $r(\phi) = a_\oplus$. For our probe, $a = \frac{9}{8}\,a_\oplus$ and $\varepsilon = \frac{1}{3}$, so $a\,(1-\varepsilon^2) = a_\oplus$, yielding $\cos\theta = 0$, meaning $\theta = \pm\frac{1}{2}\pi$. See fig. 1.

(e) We have

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} = \frac{a_\oplus\, v_\oplus^2}{4\pi^2} \ .$$

Thus,

$$\frac{a^3}{a_\oplus^3} = \left(\frac{v_\oplus\,\tau}{2\pi a_\oplus}\right)^2 = \left(\frac{\tau}{\tau_\oplus}\right)^2 \qquad \Longrightarrow \qquad \tau = \tau_\oplus \cdot \left(\frac{a}{a_\oplus}\right)^{3/2} \ .$$

With $a = \frac{9}{8}\,a_\oplus$, we obtain $\tau = 1.193\,\tau_\oplus$, with $\tau_\oplus = 1\,\mathrm{yr}$.