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Chapter 3

Systems of Particles

3.1 Work-energy theorem

Consider a system of many particles, with positions $\boldsymbol{r}_i$ and velocities $\dot{\boldsymbol{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \quad . \tag{3.1}$$

Now let's consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \, \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i \quad . \tag{3.2}$$

Here, we've used the relation

$$\frac{d}{dt}\left(\boldsymbol{A}^2\right) = 2 \, \boldsymbol{A} \cdot \frac{d\boldsymbol{A}}{dt} \quad . \tag{3.3}$$

We now invoke Newton's 2nd Law, $m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i$, to write eqn. 3.2 as $\dot{T}_i = \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i$. We integrate this equation from time t_{A} to t_{B}:

$$T_i^{\mathsf{B}} - T_i^{\mathsf{A}} = \int\limits_{t_{\mathsf{A}}}^{t_{\mathsf{B}}} dt \, \frac{dT_i}{dt} = \int\limits_{t_{\mathsf{A}}}^{t_{\mathsf{B}}} dt \, \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i \equiv \sum_i W_i^{\mathsf{A} \to \mathsf{B}} \quad , \tag{3.4}$$

where $W_i^{\mathsf{A} \to \mathsf{B}}$ is the total *work done* on particle i during its motion from state A to state B, Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{\mathsf{A} \to \mathsf{B}} = \sum_i W_i^{\mathsf{A} \to \mathsf{B}}$. Eqn. 3.4 is known as the *work-energy theorem*. It says that *In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done:* $T^{\mathsf{B}} - T^{\mathsf{A}} = W^{\mathsf{A} \to \mathsf{B}}$.

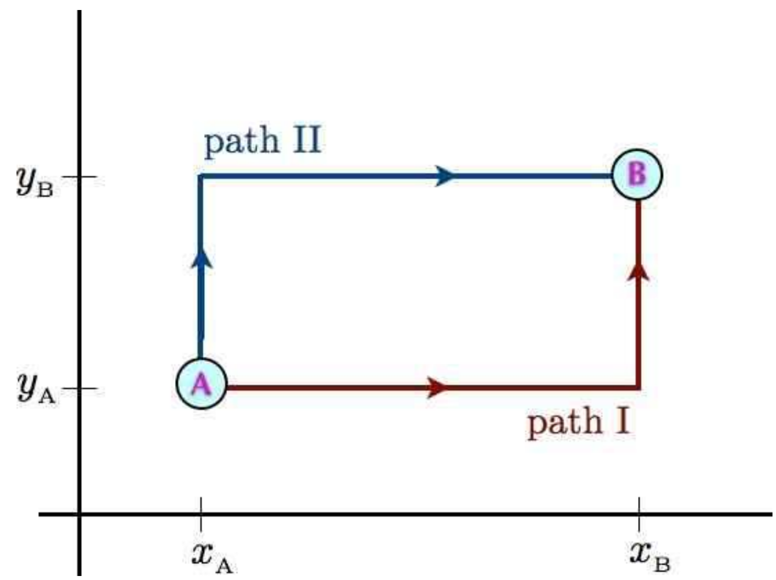

Figure 3.1: Two paths joining points A and B.

3.1.1 Conservative and nonconservative forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{\mathsf{A}\to\mathsf{B}} = \int\limits_{t_\mathsf{A}}^{t_\mathsf{B}} dt\, \boldsymbol{F}\cdot\boldsymbol{v} \quad , \tag{3.5}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$. This is the most general expression for the work done. If the force $\boldsymbol{F}$ depends only on the particle's position $\boldsymbol{r}$, we may write $d\boldsymbol{r} = \boldsymbol{v}\,dt$, and then

$$W^{\mathsf{A}\to\mathsf{B}} = \int\limits_{\boldsymbol{r}_\mathsf{A}}^{\boldsymbol{r}_\mathsf{B}} d\boldsymbol{r}\cdot\boldsymbol{F}(\boldsymbol{r}) \quad . \tag{3.6}$$

Consider now the force

$$\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\,\hat{\boldsymbol{x}} + K_2\, x\,\hat{\boldsymbol{y}} \quad , \tag{3.7}$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 3.1:

$$\begin{aligned} W^{(\mathrm{I})} &= K_1 \int\limits_{x_\mathsf{A}}^{x_\mathsf{B}} dx\, y_\mathsf{A} + K_2 \int\limits_{y_\mathsf{A}}^{y_\mathsf{B}} dy\, x_\mathsf{B} = K_1\, y_\mathsf{A}\,(x_\mathsf{B} - x_\mathsf{A}) + K_2\, x_\mathsf{B}\,(y_\mathsf{B} - y_\mathsf{A}) \\ W^{(\mathrm{II})} &= K_1 \int\limits_{x_\mathsf{A}}^{x_\mathsf{B}} dx\, y_\mathsf{B} + K_2 \int\limits_{y_\mathsf{A}}^{y_\mathsf{B}} dy\, x_\mathsf{A} = K_1\, y_\mathsf{B}\,(x_\mathsf{B} - x_\mathsf{A}) + K_2\, x_\mathsf{A}\,(y_\mathsf{B} - y_\mathsf{A}) \quad . \end{aligned} \tag{3.8}$$

Note that in general $W^{(\mathrm{I})} \neq W^{(\mathrm{II})}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(\mathrm{I}-)} - W^{(\mathrm{II})} = (K_2 - K_1)\,(x_{\mathsf{B}} - x_{\mathsf{A}})\,(y_{\mathsf{B}} - y_{\mathsf{A}}) \quad . \tag{3.9}$$

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 3.1. The reason for this is Stokes' theorem:

$$\oint\limits_{\partial\mathcal{C}} d\boldsymbol{\ell} \cdot \boldsymbol{F} = \int\limits_{\mathcal{C}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} \quad . \tag{3.10}$$

Here, $\mathcal{C}$ is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of $\mathcal{C}$, which is a closed path[1], dS is the scalar differential area element, $\hat{\boldsymbol{n}}$ is the unit normal to that differential area element, and $\boldsymbol{\nabla} \times \boldsymbol{F}$ is the curl of $\boldsymbol{F}$:

$$\begin{aligned}\boldsymbol{\nabla} \times \boldsymbol{F} &= \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{pmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}} \quad .\end{aligned} \tag{3.11}$$

For the force under consideration, $\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\, \hat{\boldsymbol{x}} + K_2\, x\, \hat{\boldsymbol{y}}$, the curl is

$$\boldsymbol{\nabla} \times \boldsymbol{F} = (K_2 - K_1)\,\hat{\boldsymbol{z}} \quad , \tag{3.12}$$

which is a constant. The RHS of eqn. 3.10 is then simply proportional to the area enclosed by $\mathcal{C}$. When we compute the work difference in eqn. 3.9, we evaluate the integral $\oint\limits_{\mathcal{C}} d\boldsymbol{\ell} \cdot \boldsymbol{F}$ along the path $\gamma_{\mathrm{II}}^{-1} \circ \gamma_{\mathrm{I}}$, which is to say path I followed by the inverse of path II. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and the integral of $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F}$ over the rectangle $\mathcal{C}$ is given by the RHS of eqn. 3.9.

When $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ everywhere in space, we can always write $\boldsymbol{F} = -\boldsymbol{\nabla} U$, where $U(\boldsymbol{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$W^{\mathsf{A}\to\mathsf{B}} = \int\limits_{\boldsymbol{r}_{\mathsf{A}}}^{\boldsymbol{r}_{\mathsf{B}}} d\boldsymbol{r} \cdot \boldsymbol{F}(\boldsymbol{r}) = -\int\limits_{\boldsymbol{r}_{\mathsf{A}}}^{\boldsymbol{r}_{\mathsf{B}}} d\boldsymbol{r} \cdot \boldsymbol{\nabla} U = U(\boldsymbol{r}_{\mathsf{A}}) - U(\boldsymbol{r}_{\mathsf{B}}) \quad . \tag{3.13}$$

The work-energy theorem then gives

$$T^{\mathsf{B}} - T^{\mathsf{A}} = U(\boldsymbol{r}_{\mathsf{A}}) - U(\boldsymbol{r}_{\mathsf{B}}) \quad , \tag{3.14}$$

which says

$$E^{\mathsf{B}} = T^{\mathsf{B}} + U(\boldsymbol{r}_{\mathsf{B}}) = T^{\mathsf{A}} + U(\boldsymbol{r}_{\mathsf{A}}) = E^{\mathsf{A}} \quad . \tag{3.15}$$

Thus, the total energy $E = T + U$ is conserved.

[1] If $\mathcal{C}$ is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if $\mathcal{C}$ is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

3.1.2 Integrating $\boldsymbol{F} = -\boldsymbol{\nabla} U$

If $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, we can compute $U(\boldsymbol{r})$ by integrating, *viz.*

$$U(\boldsymbol{r}) = U(\boldsymbol{0}) - \int\limits_{\boldsymbol{0}}^{\boldsymbol{r}} d\boldsymbol{r}' \cdot \boldsymbol{F}(\boldsymbol{r}') \quad . \tag{3.16}$$

The integral does not depend on the path chosen connecting $\boldsymbol{0}$ and $\boldsymbol{r}$. For example, we can take

$$U(x,y,z) = U(0,0,0) \; - \int\limits_{(0,0,0)}^{(x,0,0)} dx' \, F_x(x',0,0) \; - \int\limits_{(x,0,0)}^{(x,y,0)} dy' \, F_y(x,y',0) \; - \int\limits_{(z,y,0)}^{(x,y,z)} dz' \, F_z(x,y,z') \quad . \tag{3.17}$$

The constant $U(0,0,0)$ is arbitrary and impossible to determine from $\boldsymbol{F}$ alone.

As an example, consider the force

$$\boldsymbol{F}(\boldsymbol{r}) = -ky \, \hat{\boldsymbol{x}} - kx \, \hat{\boldsymbol{y}} - 4bz^3 \, \hat{\boldsymbol{z}} \quad , \tag{3.18}$$

where k and b are constants. We have

$$\begin{aligned} \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_x &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) = 0 \\ \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_y &= \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) = 0 \\ \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_z &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \quad , \end{aligned} \tag{3.19}$$

so $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ and $\boldsymbol{F}$ must be expressible as $\boldsymbol{F} = -\boldsymbol{\nabla} U$. Integrating using eqn. 3.17, we have

$$\begin{aligned} U(x,y,z) &= U(0,0,0) \; + \int\limits_{(0,0,0)}^{(x,0,0)} dx' \, k \cdot 0 \; + \int\limits_{(x,0,0)}^{(x,y,0)} dy' \, kxy' \; + \int\limits_{(z,y,0)}^{(x,y,z)} dz' \, 4bz'^3 \\ &= U(0,0,0) + kxy + bz^4 \quad . \end{aligned} \tag{3.20}$$

Another approach is to integrate the partial differential equation $\boldsymbol{\nabla} U = -\boldsymbol{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\boldsymbol{x}}$-component,

$$\frac{\partial U}{\partial x} = ky \quad . \tag{3.21}$$

Integrating, we obtain

$$U(x,y,z) = kxy + f(y,z) \quad , \tag{3.22}$$

where $f(y,z)$ is at this point an *arbitrary function* of y and z. The important thing is that it has no x-dependence, so $\partial f/\partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \quad \Longrightarrow \quad U(x,y,z) = kxy + g(x,z) \quad . \tag{3.23}$$

Finally, the z-component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \quad \Longrightarrow \quad U(x,y,z) = bz^4 + h(x,y) \quad . \tag{3.24}$$

We now equate the first two expressions:

$$kxy + f(y,z) = kxy + g(x,z) \quad . \tag{3.25}$$

Subtracting kxy from each side, we obtain the equation $f(y,z) = g(x,z)$. Since the LHS is independent of x and the RHS is independent of y, we must have

$$f(y,z) = g(x,z) = q(z) \quad , \tag{3.26}$$

where $q(z)$ is some unknown function of z. But now we invoke the final equation, to obtain

$$bz^4 + h(x,y) = kxy + q(z) \quad . \tag{3.27}$$

The only possible solution is $h(x,y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x,y,z) = C + kxy + bz^4 \quad . \tag{3.28}$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x,y,z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z. The fact that $f(y,z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y,z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y,z)$.

3.2 Conservative forces in many-particle systems

3.2.1 Kinetic and potential energies

The kinetic and potential energies are given by

$$\begin{aligned} T &= \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \\ U &= \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \quad . \end{aligned} \tag{3.29}$$

Here, $V(\boldsymbol{r})$ is the *external* (or one-body) potential, and $v(\boldsymbol{r}-\boldsymbol{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i\, \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(\mathsf{ext})} + \boldsymbol{F}_i^{(\mathsf{int})} \quad , \tag{3.30}$$

with

$$
\begin{aligned}
\boldsymbol{F}_i^{(\mathsf{ext})} &= -\frac{\partial V(\boldsymbol{r}_i)}{\partial \boldsymbol{r}_i} \\
\boldsymbol{F}_i^{(\mathsf{int})} &= -\sum_j \frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\boldsymbol{r}_i} \equiv \sum_j \boldsymbol{F}_{ij}^{(\mathsf{int})} \quad .
\end{aligned}
\tag{3.31}
$$

Here, $\boldsymbol{F}_{ij}^{(\mathsf{int})}$ is the force exerted on particle i by particle j:

$$
\boldsymbol{F}_{ij}^{(\mathsf{int})} = -\frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\partial \boldsymbol{r}_i} = -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\, v'\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \quad . \tag{3.32}
$$

Note that $\boldsymbol{F}_{ij}^{(\mathsf{int})} = -\boldsymbol{F}_{ji}^{(\mathsf{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\boldsymbol{r}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j$, in which case we may write the interparticle force as

$$
\boldsymbol{F}_{ij}^{(\mathsf{int})} = -\hat{\boldsymbol{r}}_{ij}\, v'\big(r_{ij}\big) \quad . \tag{3.33}
$$

3.2.2 Linear and angular momentum

Consider now the total momentum of the system, $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$. Its rate of change is

$$
\frac{d\boldsymbol{P}}{dt} = \sum_i \dot{\boldsymbol{p}}_i = \sum_i \boldsymbol{F}_i^{(\mathsf{ext})} + \overbrace{\sum_{i\neq j} \boldsymbol{F}_{ij}^{(\mathsf{int})}}^{\boldsymbol{F}_{ij}^{(\mathsf{int})} + \boldsymbol{F}_{ji}^{(\mathsf{int})} = 0} = \boldsymbol{F}_{\rm tot}^{(\mathsf{ext})} \quad , \tag{3.34}
$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$
\begin{aligned}
\boldsymbol{P} &= \sum_i m_i \dot{\boldsymbol{r}}_i = M\dot{\boldsymbol{R}} \\
M &= \sum_i m_i \quad \text{(total mass)} \\
\boldsymbol{R} &= \frac{\sum_i m_i\, \boldsymbol{r}_i}{\sum_i m_i} \quad \text{(center-of-mass)} \quad .
\end{aligned}
\tag{3.35}
$$

Next, consider the total angular momentum,

$$
\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \quad . \tag{3.36}
$$

The rate of change of $\boldsymbol{L}$ is then

$$
\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= \sum_i \big\{ m_i \dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i + m_i \boldsymbol{r}_i \times \ddot{\boldsymbol{r}}_i \big\} \\
&= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathsf{ext})} + \sum_{i\neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}^{(\mathsf{int})} \\
&= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathsf{ext})} + \overbrace{\tfrac{1}{2} \sum_{i\neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij}^{(\mathsf{int})}}^{\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ij}^{(\mathsf{int})} = 0} = \boldsymbol{N}_{\rm tot}^{(\mathsf{ext})} \quad .
\end{aligned}
\tag{3.37}
$$

Finally, it is useful to establish the result

$$T = \tfrac{1}{2}\sum_i m_i\,\dot{\boldsymbol{r}}_i^2 = \tfrac{1}{2}M\dot{\boldsymbol{R}}^2 + \tfrac{1}{2}\sum_i m_i\big(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{R}}\big)^2 \quad , \tag{3.38}$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the "work-energy theorem" for conservative systems,

$$\begin{aligned} 0 &= \int\limits_{\text{initial}}^{\text{final}} dE = \int\limits_{\text{initial}}^{\text{final}} dT + \int\limits_{\text{initial}}^{\text{final}} dU \\ &= T^{\mathsf{B}} - T^{\mathsf{A}} - \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i \quad , \end{aligned} \tag{3.39}$$

which is to say

$$\Delta T = T^{\mathsf{B}} - T^{\mathsf{A}} = \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i = -\Delta U \quad . \tag{3.40}$$

In other words, the total energy $E = T + U$ is conserved:

$$E = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \quad . \tag{3.41}$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz.*

$$\sum_i m_i\,\phi(\boldsymbol{r}_i) \longrightarrow \int d^3\!r\,\rho(\boldsymbol{r})\,\phi(\boldsymbol{r}) \quad , \tag{3.42}$$

where $\rho(\boldsymbol{r})$ is the mass density, and $\phi(\boldsymbol{r})$ is any function.

3.3 Scaling of Solutions for Homogeneous Potentials

3.3.1 Euler's theorem for homogeneous functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U\big(\lambda\,\boldsymbol{r}_1, \ldots, \lambda\,\boldsymbol{r}_N\big) = \lambda^k\,U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) \quad . \tag{3.43}$$

Here, k is the *degree of homogeneity* of U. Familiar examples include gravity,

$$U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) = -G\sum_{i<j} \frac{m_i\,m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \quad ; \quad k = -1 \quad , \tag{3.44}$$

and the harmonic oscillator,

$$U\big(q_1,\ldots,q_n\big)=\tfrac{1}{2}\sum_{\sigma,\sigma'} V_{\sigma\sigma'}\,q_\sigma\,q_{\sigma'} \quad ; \quad k=+2 \quad . \tag{3.45}$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1,\ldots,x_n)$ is homogeneous:

$$H(\lambda\,x_1,\ldots,\lambda\,x_n)=\lambda^k\,H(x_1,\ldots,x_n) \quad . \tag{3.46}$$

Then

$$\frac{d}{d\lambda}\bigg|_{\lambda=1} H\big(\lambda\,x_1,\ldots,\lambda\,x_n\big)=\sum_{i=1}^n x_i\,\frac{\partial H}{\partial x_i}=kH \quad . \tag{3.47}$$

3.3.2 Scaled equations of motion

Now suppose the we rescale distances and times, defining

$$\boldsymbol{r}_i=\alpha\,\tilde{\boldsymbol{r}}_i \qquad , \qquad t=\beta\,\tilde{t} \quad . \tag{3.48}$$

Then

$$\frac{d\boldsymbol{r}_i}{dt}=\frac{\alpha}{\beta}\,\frac{d\tilde{\boldsymbol{r}}_i}{d\tilde{t}} \qquad , \qquad \frac{d^2\boldsymbol{r}_i}{dt^2}=\frac{\alpha}{\beta^2}\,\frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} \quad . \tag{3.49}$$

The force $\boldsymbol{F}_i$ is given by

$$\begin{aligned}\boldsymbol{F}_i&=-\frac{\partial}{\partial\boldsymbol{r}_i}\,U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big)\\&=-\frac{\partial}{\partial(\alpha\tilde{\boldsymbol{r}}_i)}\,\alpha^k\,U\big(\tilde{\boldsymbol{r}}_1,\ldots,\tilde{\boldsymbol{r}}_N\big)\equiv\alpha^{k-1}\,\widetilde{\boldsymbol{F}}_i \quad ,\end{aligned} \tag{3.50}$$

where $\widetilde{\boldsymbol{F}}_i=\partial U\big(\tilde{\boldsymbol{r}}_1,\ldots,\tilde{\boldsymbol{r}}_N\big)/\partial\tilde{\boldsymbol{r}}_i$. Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2}\,m_i\,\frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2}=\alpha^{k-1}\,\widetilde{\boldsymbol{F}}_i \quad . \tag{3.51}$$

If we choose β such that

$$\frac{\alpha}{\beta^2}=\alpha^{k-1} \quad \Rightarrow \quad \beta=\alpha^{1-\frac{1}{2}k} \quad , \tag{3.52}$$

then the equation of motion is invariant under the rescaling transformation, *i.e.*

$$m_i\,\frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2}=\widetilde{\boldsymbol{F}}_i \quad . \tag{3.53}$$

This means that if $\big\{\boldsymbol{r}_i(t)\big\}$ is a solution to the equations of motion, then so is $\big\{\alpha\,\boldsymbol{r}_i(\beta t)\big\}$. This gives us an entire one-parameter family of solutions, for all real positive α. with $\beta=\alpha^{1-\frac{1}{2}k}$.

We see that if $\boldsymbol{r}_i(t)$ is periodic with period T, then $\boldsymbol{r}_i(t;\alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k}\,T$. Furthermore, if L is a length scale associated with an orbit $\boldsymbol{r}_i(t)$, such as the distance of closest approach, then we have the following relation between the ratios of the time and length scales:

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} \quad . \tag{3.54}$$

Velocities, energies and angular momenta scale accordingly Thus

$$[v] = \frac{L}{T} \quad \Rightarrow \quad \frac{v'}{v} = \frac{L'}{L}\bigg/\frac{T'}{T} = \alpha^{\frac{1}{2}k} \tag{3.55}$$

and

$$[E] = \frac{ML^2}{T^2} \quad \Rightarrow \quad \frac{E'}{E} = \left(\frac{L'}{L}\right)^2\bigg/\left(\frac{T'}{T}\right)^2 = \alpha^k \tag{3.56}$$

and

$$[\boldsymbol{L}] = \frac{ML^2}{T} \quad \Rightarrow \quad \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \left(\frac{L'}{L}\right)^2\bigg/\frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)} \quad . \tag{3.57}$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t;\alpha) = \alpha\, q_\sigma(t) \quad . \tag{3.58}$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\boldsymbol{r}(t) \longrightarrow \boldsymbol{r}(t;\alpha) = \alpha\,\boldsymbol{r}\big(\alpha^{-3/2}\,t\big) \quad . \tag{3.59}$$

Thus, $r^3 \propto t^2$, *i.e.*

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2 \quad , \tag{3.60}$$

also known as Kepler's Third Law.

3.4 Appendix : Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \ldots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \ldots, q_d\}$, which are related to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu(x_1, \ldots, x_d) \quad . \tag{3.61}$$

In general these are nonlinear equations.

Let $\hat{\boldsymbol{e}}_i^0 = \hat{\boldsymbol{x}}_i$ be the Cartesian set of orthonormal unit vectors, and define $\hat{\boldsymbol{e}}_\mu$ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\boldsymbol{s} = \sum_{i=1}^{d} \hat{\boldsymbol{e}}_i^0 \, dx_i = \sum_{\mu=1}^{d} \hat{\boldsymbol{e}}_\mu \, h_\mu(q) \, dq_\mu \quad , \tag{3.62}$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν. Finding the coefficient of dq_μ then gives

$$h_\mu(q) \, \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^{d} \frac{\partial x_i}{\partial q_\mu} \, \hat{\boldsymbol{e}}_i^0 \qquad \Rightarrow \quad \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^{d} M_{\mu\, i} \, \hat{\boldsymbol{e}}_i^0 \quad , \tag{3.63}$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)} \, \frac{\partial x_i}{\partial q_\mu} \quad . \tag{3.64}$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{\boldsymbol{e}}_\mu \cdot \hat{\boldsymbol{e}}_\nu = \big(M M^{\rm t}\big)_{\mu\nu} = \frac{1}{h_\mu(q) \, h_\nu(q)} \sum_{i=1}^{d} \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} \quad . \tag{3.65}$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^{d} \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} = h_\mu^2(q) \, \delta_{\mu\nu} \quad . \tag{3.66}$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^{d} \left(\frac{\partial x_i}{\partial q_\mu}\right)^2} \quad . \tag{3.67}$$

Note that

$$(d\boldsymbol{s})^2 = \sum_{\mu=1}^{d} h_\mu^2(q) \, (dq_\mu)^2 \quad . \tag{3.68}$$

For general coordinate systems, which are not necessarily orthogonal, we have

$$(d\boldsymbol{s})^2 = \sum_{\mu,\nu=1}^{d} g_{\mu\nu}(q) \, dq_\mu \, dq_\nu \quad , \tag{3.69}$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

3.4.1 Example : spherical coordinates

Consider spherical coordinates (ρ, θ, ϕ):

$$x = \rho \, \sin\theta \, \cos\phi \quad , \quad y = \rho \, \sin\theta \, \sin\phi \quad , \quad z = \rho \, \cos\theta \quad . \tag{3.70}$$

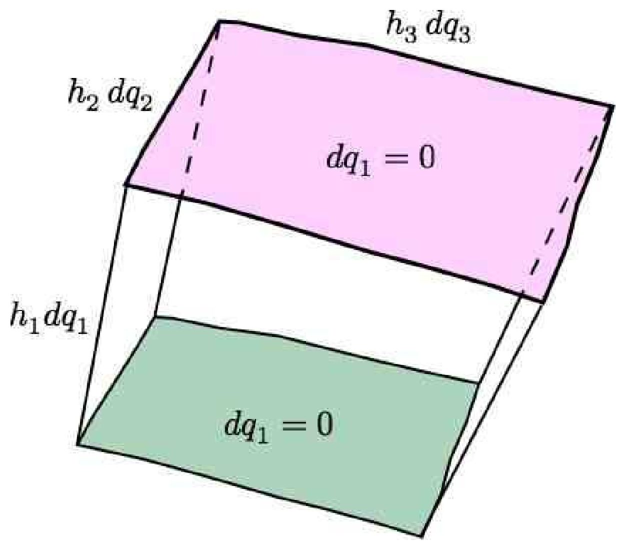

Figure 3.2: Volume element Ω for computing divergences.

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2 \, \sin^2\!\theta \quad . \tag{3.71}$$

Thus,

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \rho\, \hat{\boldsymbol{\theta}}\, d\theta + \rho\, \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi \quad . \tag{3.72}$$

3.4.2 Vector calculus : grad, div, curl

Here we restrict our attention to $d = 3$. The gradient $\boldsymbol{\nabla} U$ of a function $U(q)$ is defined by

$$\begin{aligned} dU &= \frac{\partial U}{\partial q_1}\, dq_1 + \frac{\partial U}{\partial q_2}\, dq_2 + \frac{\partial U}{\partial q_3}\, dq_3 \\ &\equiv \boldsymbol{\nabla} U \cdot d\boldsymbol{s} \quad . \end{aligned} \tag{3.73}$$

Thus,

$$\boldsymbol{\nabla} = \frac{\hat{\boldsymbol{e}}_1}{h_1(q)}\, \frac{\partial}{\partial q_1} + \frac{\hat{\boldsymbol{e}}_2}{h_2(q)}\, \frac{\partial}{\partial q_2} + \frac{\hat{\boldsymbol{e}}_3}{h_3(q)}\, \frac{\partial}{\partial q_3} \quad . \tag{3.74}$$

For the divergence, we use the divergence theorem, and we appeal to fig. 3.2:

$$\int\limits_{\Omega} dV\, \boldsymbol{\nabla} \cdot \boldsymbol{A} = \int\limits_{\partial\Omega} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{A} \quad , \tag{3.75}$$

where Ω is a region of three-dimensional space and $\partial\Omega$ is its closed two-dimensional boundary. The LHS of this equation is

$$\text{LHS} = \boldsymbol{\nabla} \cdot \boldsymbol{A} \cdot (h_1\, dq_1)\, (h_2\, dq_2)\, (h_3\, dq_3) \quad . \tag{3.76}$$

The RHS is

$$\begin{aligned}\mathrm{RHS} &= A_1\, h_2\, h_3 \Big|_{q_1}^{q_1+dq_1} dq_2\, dq_3 + A_2\, h_1\, h_3 \Big|_{q_2}^{q_2+dq_2} dq_1\, dq_3 + A_3\, h_1\, h_2 \Big|_{q_3}^{q_1+dq_3} dq_1\, dq_2 \\ &= \bigg[\frac{\partial}{\partial q_1}\big(A_1\, h_2\, h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\, h_1\, h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\, h_1\, h_2\big)\bigg]\, dq_1\, dq_2\, dq_3 \quad .\end{aligned} \tag{3.77}$$

We therefore conclude

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{h_1\, h_2\, h_3}\bigg[\frac{\partial}{\partial q_1}\big(A_1\, h_2\, h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\, h_1\, h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\, h_1\, h_2\big)\bigg] \quad . \tag{3.78}$$

To obtain the curl $\boldsymbol{\nabla}\times\boldsymbol{A}$, we use Stokes' theorem again,

$$\int\limits_{\Sigma} dS\, \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = \oint\limits_{\partial\Sigma} d\boldsymbol{\ell}\cdot\boldsymbol{A} \quad , \tag{3.79}$$

where Σ is a two-dimensional region of space and $\partial\Sigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2\, dq_2$ and $h_3\, dq_3$. The LHS of the above equation is

$$\mathrm{LHS} = \hat{\boldsymbol{e}}_1\cdot\boldsymbol{\nabla}\times\boldsymbol{A}\,(h_2\, dq_2)\,(h_3\, dq_3) \quad . \tag{3.80}$$

The RHS is

$$\begin{aligned}\mathrm{RHS} &= A_3\, h_3 \Big|_{q_2}^{q_2+dq_2} dq_3 - A_2\, h_2 \Big|_{q_3}^{q_3+dq_3} dq_2 \\ &= \bigg[\frac{\partial}{\partial q_2}\big(A_3\, h_3\big) - \frac{\partial}{\partial q_3}\big(A_2\, h_2\big)\bigg] dq_2\, dq_3 \quad .\end{aligned} \tag{3.81}$$

Therefore

$$(\boldsymbol{\nabla}\times\boldsymbol{A})_1 = \frac{1}{h_2\, h_3}\bigg(\frac{\partial(h_3\, A_3)}{\partial q_2} - \frac{\partial(h_2\, A_2)}{\partial q_3}\bigg) \quad . \tag{3.82}$$

This is one component of the full result

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{h_1\, h_2\, h_3}\,\mathrm{det}\begin{pmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1\, A_1 & h_2\, A_2 & h_3\, A_3 \end{pmatrix} \quad . \tag{3.83}$$

The Laplacian of a scalar function U is given by

$$\begin{aligned}\nabla^2 U &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} U \\ &= \frac{1}{h_1\, h_2\, h_3}\bigg\{\frac{\partial}{\partial q_1}\bigg(\frac{h_2\, h_3}{h_1}\,\frac{\partial U}{\partial q_1}\bigg) + \frac{\partial}{\partial q_2}\bigg(\frac{h_1\, h_3}{h_2}\,\frac{\partial U}{\partial q_2}\bigg) + \frac{\partial}{\partial q_3}\bigg(\frac{h_1\, h_2}{h_3}\,\frac{\partial U}{\partial q_3}\bigg)\bigg\} \quad .\end{aligned} \tag{3.84}$$

Rectangular coordinates

In *rectangular* coordinates (x, y, z), we have

$$h_x = h_y = h_z = 1 \quad . \tag{3.85}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy + \hat{\boldsymbol{z}}\, dz \tag{3.86}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \quad . \tag{3.87}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{x}}\, \frac{\partial U}{\partial x} + \hat{\boldsymbol{y}}\, \frac{\partial U}{\partial y} + \hat{\boldsymbol{z}}\, \frac{\partial U}{\partial z} \quad . \tag{3.88}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad . \tag{3.89}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}} \quad . \tag{3.90}$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \quad . \tag{3.91}$$

Cylindrical coordinates

In *cylindrical* coordinates (ρ, ϕ, z), we have

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\, \cos\phi + \hat{\boldsymbol{y}}\, \sin\phi \quad , \quad \hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}}\, \cos\phi - \hat{\boldsymbol{\phi}}\, \sin\phi \quad , \quad d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\, d\phi \tag{3.92}$$

and

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\, \sin\phi + \hat{\boldsymbol{y}}\, \cos\phi \quad , \quad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\, \sin\phi + \hat{\boldsymbol{\phi}}\, \cos\phi \quad , \quad d\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}\, d\phi \quad . \tag{3.93}$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 \quad . \tag{3.94}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \hat{\boldsymbol{\phi}}\, \rho\, d\phi + \hat{\boldsymbol{z}}\, dz \tag{3.95}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \quad . \tag{3.96}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{\rho}}\, \frac{\partial U}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho}\, \frac{\partial U}{\partial \phi} + \hat{\boldsymbol{z}}\, \frac{\partial U}{\partial z} \quad . \tag{3.97}$$

The divergence is

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=\frac{1}{\rho}\,\frac{\partial(\rho\,A_\rho)}{\partial\rho}+\frac{1}{\rho}\,\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}\quad. \tag{3.98}$$

The curl is

$$\boldsymbol{\nabla}\times\boldsymbol{A}=\left(\frac{1}{\rho}\,\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}}+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}}+\left(\frac{1}{\rho}\,\frac{\partial(\rho A_\phi)}{\partial\rho}-\frac{1}{\rho}\,\frac{\partial A_\rho}{\partial\phi}\right)\hat{\boldsymbol{z}}\quad. \tag{3.99}$$

The Laplacian is

$$\nabla^2 U=\frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial U}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2 U}{\partial\phi^2}+\frac{\partial^2 U}{\partial z^2}\quad. \tag{3.100}$$

Spherical coordinates

In *spherical* coordinates (r,θ,ϕ), we have

$$\begin{aligned}\hat{\boldsymbol{r}}&=\hat{\boldsymbol{x}}\sin\theta\cos\phi+\hat{\boldsymbol{y}}\sin\theta\sin\phi+\hat{\boldsymbol{z}}\sin\theta\\ \hat{\boldsymbol{\theta}}&=\hat{\boldsymbol{x}}\cos\theta\cos\phi+\hat{\boldsymbol{y}}\cos\theta\sin\phi-\hat{\boldsymbol{z}}\cos\theta\\ \hat{\boldsymbol{\phi}}&=-\hat{\boldsymbol{x}}\sin\phi+\hat{\boldsymbol{y}}\cos\phi\quad,\end{aligned} \tag{3.101}$$

for which

$$\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\phi}}\quad,\quad\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}=\hat{\boldsymbol{r}}\quad,\quad\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{r}}=\hat{\boldsymbol{\theta}}\quad. \tag{3.102}$$

The inverse is

$$\begin{aligned}\hat{\boldsymbol{x}}&=\hat{\boldsymbol{r}}\sin\theta\cos\phi+\hat{\boldsymbol{\theta}}\cos\theta\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi\\ \hat{\boldsymbol{y}}&=\hat{\boldsymbol{r}}\sin\theta\sin\phi+\hat{\boldsymbol{\theta}}\cos\theta\sin\phi+\hat{\boldsymbol{\phi}}\cos\phi\\ \hat{\boldsymbol{z}}&=\hat{\boldsymbol{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\quad.\end{aligned} \tag{3.103}$$

The differential relations are

$$\begin{aligned}d\hat{\boldsymbol{r}}&=\hat{\boldsymbol{\theta}}\,d\theta+\sin\theta\,\hat{\boldsymbol{\phi}}\,d\phi\\ d\hat{\boldsymbol{\theta}}&=-\hat{\boldsymbol{r}}\,d\theta+\cos\theta\,\hat{\boldsymbol{\phi}}\,d\phi\\ d\hat{\boldsymbol{\phi}}&=-\big(\sin\theta\,\hat{\boldsymbol{r}}+\cos\theta\,\hat{\boldsymbol{\theta}}\big)\,d\phi\end{aligned} \tag{3.104}$$

The metric is given in terms of

$$h_r=1\quad,\quad h_\theta=r\quad,\quad h_\phi=r\,\sin\theta\quad. \tag{3.105}$$

Thus

$$d\boldsymbol{s}=\hat{\boldsymbol{r}}\,dr+\hat{\boldsymbol{\theta}}\,r\,d\theta+\hat{\boldsymbol{\phi}}\,r\sin\theta\,d\phi \tag{3.106}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2=\dot r^2+r^2\dot\theta^2+r^2\sin^2\!\theta\,\dot\phi^2\quad. \tag{3.107}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{r}}\,\frac{\partial U}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\,\frac{\partial U}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\,\frac{\partial U}{\partial \phi} \quad . \tag{3.108}$$

The divergence is

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^2}\,\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\,\frac{\partial A_\phi}{\partial\phi} \quad . \tag{3.109}$$

The curl is

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{r\sin\theta}&\left(\frac{\partial(\sin\theta\, A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\,\frac{\partial A_r}{\partial\phi} - \frac{\partial(r A_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \quad .\end{aligned} \tag{3.110}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2}\,\frac{\partial}{\partial r}\!\left(r^2\,\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\!\left(\sin\theta\,\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\!\theta}\,\frac{\partial^2 U}{\partial\phi^2} \quad . \tag{3.111}$$

Kinetic energy

Note the form of the kinetic energy of a point particle:

$$\begin{aligned} T = \tfrac{1}{2}m\left(\frac{d\boldsymbol{s}}{dt}\right)^2 &= \tfrac{1}{2}m\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) && \text{(3D Cartesian)} \\ &= \tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2\big) && \text{(2D polar)} \\ &= \tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\big) && \text{(3D cylindrical)} \\ &= \tfrac{1}{2}m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\!\theta\,\dot{\phi}^2\big) && \text{(3D polar)} \quad . \end{aligned} \tag{3.112}$$