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Chapter 6

Linearized Dynamics of Coupled Oscillations

6.1 Basic Objective

Our basic objective in studying small coupled oscillations is to expand the equations of motion to linear order in the n generalized coordinates about a stable equilibrium configuration. This yields a set of n coupled second order differential equations that is both *linear* and *homogeneous*. Such a system may then be solved by elementary linear algebraic means. The general solution may then be written as a sum over n *normal mode oscillations*, each of which oscillates at a particular *eigenfrequency* ω_j, with $j \in \{1, \ldots, n\}$. The set of eigenfrequencies is determined by the form of the linearized equations of motion. The n normal mode amplitudes and n normal mode phase shifts are determined by the $2n$ initial conditions on the generalized coordinates and velocities.

6.2 Euler-Lagrange Equations of Motion

We assume, for a set of n generalized coordinates $\{q_1, \ldots, q_n\}$, that the kinetic energy is a quadratic function of the velocities,

$$T = \tfrac{1}{2}\, T_{\sigma\sigma'}(q_1, \ldots, q_n)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} \quad , \tag{6.1}$$

where the sum on σ and σ' from 1 to n is implied. For example, expressed in terms of polar coordinates (r, θ, ϕ), the matrix $T_{\sigma\sigma'}$ is

$$T_{\sigma\sigma'}(r,\theta,\phi) = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\!\theta \end{pmatrix} \quad \Longrightarrow \quad T = \tfrac{1}{2}\, m\big(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\!\theta\, \dot{\phi}^2\big) \quad . \tag{6.2}$$

The potential $U(q_1, \ldots, q_n)$ is assumed to be a function of the generalized coordinates alone: $U = U(q)$. A more general formulation of the problem of small oscillations is given in the appendix, section 6.8.

The generalized momenta are

$$p_\sigma = \frac{\partial L}{\partial \dot q_\sigma} = T_{\sigma\sigma'}\,\dot q_{\sigma'} \quad , \tag{6.3}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{1}{2}\,\frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\,\dot q_{\sigma'}\,\dot q_{\sigma''} - \frac{\partial U}{\partial q_\sigma} \quad . \tag{6.4}$$

The Euler-Lagrange equations are then $\dot p_\sigma = F_\sigma$, or

$$T_{\sigma\sigma'}\,\ddot q_{\sigma'} + \bigg(\frac{\partial T_{\sigma\sigma'}}{\partial q_{\sigma''}} - \frac{1}{2}\,\frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\bigg)\,\dot q_{\sigma'}\,\dot q_{\sigma''} = -\frac{\partial U}{\partial q_\sigma} \tag{6.5}$$

which is a set of coupled nonlinear second order ODEs. Here we are using the Einstein 'summation convention', where we automatically sum over any and all repeated indices.

6.3 Expansion about Static Equilibrium

Small oscillation theory begins with the identification of a static equilibrium $\{\bar q_1,\ldots,\bar q_n\}$, which satisfies the n nonlinear equations

$$\frac{\partial U}{\partial q_\sigma}\bigg|_{q=\bar q} = 0 \quad . \tag{6.6}$$

Once an equilibrium is found (note that there may be more than one static equilibrium), we expand about this equilibrium, writing

$$q_\sigma \equiv \bar q_\sigma + \eta_\sigma \quad . \tag{6.7}$$

The coordinates $\{\eta_1,\ldots,\eta_n\}$ represent the *displacements relative to equilibrium.*

We next expand the Lagrangian to quadratic order in the generalized displacements, yielding

$$L = \tfrac{1}{2}\mathsf{T}_{\sigma\sigma'}\,\dot\eta_\sigma\,\dot\eta_{\sigma'} - \tfrac{1}{2}\mathsf{V}_{\sigma\sigma'}\,\eta_\sigma\eta_{\sigma'} \quad , \tag{6.8}$$

where

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot q_\sigma\,\partial\dot q_{\sigma'}}\bigg|_{q=\bar q} \qquad , \qquad \mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q=\bar q} \quad . \tag{6.9}$$

Writing $\boldsymbol{\eta}^{\mathsf{t}}$ for the row-vector $(\eta_1,\ldots,\eta_n)$, we may suppress indices and write

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\eta}}^{\mathsf{t}}\,\mathsf{T}\,\dot{\boldsymbol{\eta}} - \tfrac{1}{2}\,\boldsymbol{\eta}^{\mathsf{t}}\,\mathsf{V}\,\boldsymbol{\eta} \quad , \tag{6.10}$$

where T and V are the constant matrices of eqn. 6.9.

6.4 Method of Small Oscillations

The idea behind the method of small oscillations is to effect a coordinate transformation from the generalized displacements $\boldsymbol{\eta}$ to a new set of coordinates $\boldsymbol{\xi}$, which render the Lagrangian particularly simple. All that is required is a linear transformation,

$$\eta_\sigma = \mathsf{A}_{\sigma i}\,\xi_i \quad , \tag{6.11}$$

where both σ and i run from 1 to n. The $n \times n$ matrix $\mathsf{A}_{\sigma i}$ is known as the *modal matrix*. With the substitution $\boldsymbol{\eta} = \mathsf{A}\,\boldsymbol{\xi}$ (hence $\boldsymbol{\eta}^{\mathsf{t}} = \boldsymbol{\xi}^{\mathsf{t}}\,\mathsf{A}^{\mathsf{t}}$, where $\mathsf{A}^{\mathsf{t}}_{i\sigma} = \mathsf{A}_{\sigma i}$ is the matrix transpose), we have

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\xi}}^{\mathsf{t}}\,\mathsf{A}^{\mathsf{t}}\,\mathsf{T}\,\mathsf{A}\,\dot{\boldsymbol{\xi}} - \tfrac{1}{2}\,\boldsymbol{\xi}^{\mathsf{t}}\,\mathsf{A}^{\mathsf{t}}\,\mathsf{V}\,\mathsf{A}\,\boldsymbol{\xi} \quad . \tag{6.12}$$

We now choose the matrix A such that

$$\begin{aligned} &\mathsf{A}^{\mathsf{t}}\,\mathsf{T}\,\mathsf{A} = \mathbb{I} \\ &\mathsf{A}^{\mathsf{t}}\,\mathsf{V}\,\mathsf{A} = \mathsf{diag}\big(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\big) \quad . \end{aligned} \tag{6.13}$$

With this choice of A, the Lagrangian decouples:

$$L = \tfrac{1}{2}\sum_{i=1}^{n}\Big(\dot{\xi}_i^2 - \omega_i^2\,\xi_i^2\Big) \quad , \tag{6.14}$$

with the solution

$$\xi_i(t) = C_i\cos(\omega_i\,t) + D_i\sin(\omega_i\,t) \quad , \tag{6.15}$$

where $\{C_1,\ldots,C_n\}$ and $\{D_1,\ldots,D_n\}$ are $2n$ constants of integration, determined by the $2n$ initial conditions on $\boldsymbol{\eta}(0)$ and $\dot{\boldsymbol{\eta}}(0)$, and where there is no implied sum on i. Note that

$$\boldsymbol{\xi} = \mathsf{A}^{-1}\boldsymbol{\eta} = \mathsf{A}^{\mathsf{t}}\,\mathsf{T}\boldsymbol{\eta} \quad . \tag{6.16}$$

In terms of the original generalized displacements, the solution is

$$\eta_\sigma(t) = \sum_{i=1}^{n}\mathsf{A}_{\sigma i}\,\Big\{C_i\cos(\omega_i t) + D_i\sin(\omega_i t)\Big\} \quad , \tag{6.17}$$

and the constants of integration are linearly related to the initial generalized displacements and generalized velocities:

$$\begin{aligned} C_i &= \mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\eta_{\sigma'}(0) \\ D_i &= \omega_i^{-1}\mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'}(0) \quad , \end{aligned} \tag{6.18}$$

again with no implied sum on i on the RHS of the second equation, and where we have used $\mathsf{A}^{-1} = \mathsf{A}^{\mathsf{t}}\,\mathsf{T}$, from eqn. 6.13. (The implied sums in eqn. 6.18 are over σ and σ'.)

If all the generalized coordinates have units of length, *i.e.* $[q_\sigma] = L$, then

$$[\mathsf{T}_{\sigma\sigma'}] = M \quad , \quad [\mathsf{V}_{\sigma\sigma'}] = MT^{-2} \quad , \quad [A_{\sigma i}] = M^{-1/2} \quad , \quad [\xi_i] = M^{1/2}L \quad . \tag{6.19}$$

Can you really just choose an A so that both of eqns. 6.13 hold?

Yes.

Er...care to elaborate?

Both T and V are symmetric matrices. Aside from that, there is no special relation between them. In particular, they need not commute, hence they do not necessarily share any eigenvectors. Nevertheless, they may be simultaneously diagonalized as per eqns. 6.13. Here's why:

- Since T is symmetric, it can be diagonalized by an orthogonal transformation. That is, there exists a matrix $\mathsf{O}_1 \in \mathrm{O}(n)$ such that

$$\mathsf{O}_1^{\mathsf{t}}\,\mathsf{T}\,\mathsf{O}_1 = \mathsf{D} \quad , \tag{6.20}$$

  where D is diagonal.

- We may safely assume that T is positive definite. Otherwise the kinetic energy can become arbitrarily negative, which is unphysical. Therefore, one may form the matrix $\mathsf{D}^{-1/2}$ which is the diagonal matrix whose entries are the inverse square roots of the corresponding entries of D. Consider the linear transformation $\mathsf{O}_1\mathsf{D}^{-1/2}$. Its effect on T is

$$\mathsf{D}^{-1/2}\,\overbrace{\mathsf{O}_1^{\mathsf{t}}\,\mathsf{T}\,\mathsf{O}_1}^{\mathsf{D}}\,\mathsf{D}^{-1/2} = \mathbb{I} \quad . \tag{6.21}$$

- Since O_1 and D are wholly derived from T, the only thing we know about

$$\widetilde{\mathsf{V}} \equiv \mathsf{D}^{-1/2}\,\mathsf{O}_1^{\mathsf{t}}\,\mathsf{V}\,\mathsf{O}_1\,\mathsf{D}^{-1/2} \tag{6.22}$$

  is that it is explicitly a symmetric matrix. Therefore, it may be diagonalized by some orthogonal matrix $\mathsf{O}_2 \in \mathrm{O}(n)$. As T has already been transformed to the identity, the additional orthogonal transformation has no effect there. Thus, we have shown that there exist orthogonal matrices O_1 and O_2 such that

$$\begin{aligned} \mathsf{O}_2^{\mathsf{t}}\,\mathsf{D}^{-1/2}\,\mathsf{O}_1^{\mathsf{t}}\,\mathsf{T}\,\mathsf{O}_1\,\mathsf{D}^{-1/2}\,\mathsf{O}_2 &= \mathbb{I} \\ \mathsf{O}_2^{\mathsf{t}}\,\mathsf{D}^{-1/2}\,\mathsf{O}_1^{\mathsf{t}}\,\mathsf{V}\,\mathsf{O}_1\,\mathsf{D}^{-1/2}\,\mathsf{O}_2 &= \mathsf{diag}(\omega_1^2,\ldots,\omega_n^2) \quad . \end{aligned} \tag{6.23}$$

  All that remains is to identify the modal matrix $\mathsf{A} = \mathsf{O}_1\mathsf{D}^{-1/2}\,\mathsf{O}_2$.

Note that it is *not possible* to simultaneously diagonalize *three* symmetric matrices in general.

6.4.1 Finding the modal matrix

While the above proof allows one to construct A by finding the two orthogonal matrices O_1 and O_2, such a procedure is extremely cumbersome. It would be much more convenient if A could be determined in one fell swoop. Fortunately, this is possible.

We start with the equations of motion, $\mathsf{T}\ddot{\boldsymbol{\eta}} + \mathsf{V}\boldsymbol{\eta} = 0$. In component notation, we have

$$\mathsf{T}_{\sigma\sigma'}\,\ddot{\eta}_{\sigma'} + \mathsf{V}_{\sigma\sigma'}\,\eta_{\sigma'} = 0 \quad . \tag{6.24}$$

We now assume that $\boldsymbol{\eta}(t)$ oscillates with a single frequency ω, *i.e.* $\eta_\sigma(t) = \psi_\sigma\, e^{-i\omega t}$. This results in a set of linear algebraic equations for the components ψ_σ:

$$\big(\omega^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi_{\sigma'} = 0 \quad . \tag{6.25}$$

These are n equations in n unknowns: one for each value of $\sigma = 1, \ldots, n$. Because the equations are homogeneous and linear, there is always a trivial solution $\boldsymbol{\psi} = 0$. In fact one might think this is the only solution, since

$$\big(\omega^2\,\mathsf{T} - \mathsf{V}\big)\,\boldsymbol{\psi} = 0 \qquad \stackrel{?}{\Longrightarrow} \qquad \boldsymbol{\psi} = \big(\omega^2\,\mathsf{T} - \mathsf{V}\big)^{-1}\,0 = 0 \quad . \tag{6.26}$$

However, this fails when the matrix $\omega^2\,\mathsf{T} - \mathsf{V}$ is defective[1], *i.e.* when

$$\mathsf{det}\big(\omega^2\,\mathsf{T} - \mathsf{V}\big) = 0 \quad . \tag{6.27}$$

Since T and V are of rank n, the above determinant yields an $n^{\rm th}$ order polynomial in ω^2, whose n roots are the desired squared eigenfrequencies $\{\omega_1^2\,, \ldots,\, \omega_n^2\}$.

Once the n eigenfrequencies are obtained, the modal matrix is constructed as follows. Solve the equations

$$\sum_{\sigma'=1}^{n} \big(\omega_i^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'} = 0 \tag{6.28}$$

which are a set of $(n-1)$ linearly independent equations among the n components of the eigenvector $\boldsymbol{\psi}^{(i)}$. That is, there are n equations ($\sigma = 1, \ldots, n$), but one linear dependency since $\mathsf{det}\,(\omega_i^2\,\mathsf{T} - \mathsf{V}) = 0$. The eigenvectors may be chosen to satisfy a generalized orthogonality relationship,

$$\psi^{(i)}_\sigma\,\mathsf{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'} = \delta_{ij} \quad . \tag{6.29}$$

To see this, let us duplicate eqn. 6.28, replacing i with j, and multiply both equations as follows:

$$\begin{aligned} \psi^{(j)}_\sigma \times \big(\omega_i^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'} &= 0 \\ \psi^{(i)}_\sigma \times \big(\omega_j^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi^{(j)}_{\sigma'} &= 0 \quad . \end{aligned} \tag{6.30}$$

Using the symmetry of T and V, upon subtracting these equations we obtain

$$(\omega_i^2 - \omega_j^2)\sum_{\sigma,\sigma'=1}^{n} \psi^{(i)}_\sigma\,\mathsf{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'} = 0 \quad , \tag{6.31}$$

[1] The label *defective* has a distastefully negative connotation. In modern parlance, we should instead refer to such a matrix as *determinantally challenged*.

where the sums on i and j have been made explicit. This establishes that eigenvectors $\boldsymbol{\psi}^{(i)}$ and $\boldsymbol{\psi}^{(j)}$ corresponding to distinct eigenvalues $\omega_i^2 \neq \omega_j^2$ are orthogonal: $(\boldsymbol{\psi}^{(i)})^{\mathsf{t}}\, \mathsf{T}\, \boldsymbol{\psi}^{(j)} = 0$. For degenerate eigenvalues, the eigenvectors are not *a priori* orthogonal, but they may be orthogonalized via application of the Gram-Schmidt procedure. The remaining degrees of freedom - one for each eigenvector – are fixed by imposing the condition of normalization:

$$\psi^{(i)}_\sigma \to \psi^{(i)}_\sigma \Big/ \sqrt{\psi^{(i)}_\mu\, \mathsf{T}_{\mu\mu'}\, \psi^{(i)}_{\mu'}} \qquad \Longrightarrow \qquad \psi^{(i)}_\sigma\, \mathsf{T}_{\sigma\sigma'}\, \psi^{(j)}_{\sigma'} = \delta_{ij} \quad . \tag{6.32}$$

The modal matrix is just the matrix of eigenvectors: $\mathsf{A}_{\sigma i} = \psi^{(i)}_\sigma$.

With the eigenvectors $\psi^{(i)}_\sigma$ thusly normalized, we have

$$\begin{aligned} 0 &= \psi^{(i)}_\sigma \left(\omega_j^2\, \mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\right) \psi^{(j)}_{\sigma'} \\ &= \omega_j^2\, \delta_{ij} - \psi^{(i)}_\sigma\, \mathsf{V}_{\sigma\sigma'}\, \psi^{(j)}_{\sigma'} \quad , \end{aligned} \tag{6.33}$$

with no sum on j. This establishes the result

$$\mathsf{A}^{\mathsf{t}}\, \mathsf{V}\, \mathsf{A} = \mathsf{diag}\big(\omega_1^2\,,\, \ldots\,,\, \omega_n^2\big) \quad . \tag{6.34}$$

Recall the relation $\eta_\sigma = \mathsf{A}_{\sigma i}\, \xi_i$ between the generalized displacements η_σ and the normal coordinates ξ_i. We can invert this relation to obtain

$$\xi_i = \mathsf{A}^{-1}_{i\sigma}\, \eta_\sigma = \mathsf{A}^{\mathsf{t}}_{i\sigma} \mathsf{T}_{\sigma\sigma'}\, \eta_{\sigma'} \quad . \tag{6.35}$$

Here we have used the result $\mathsf{A}^{\mathsf{t}}\, \mathsf{T}\, \mathsf{A} = 1$ to write

$$\mathsf{A}^{-1} = \mathsf{A}^{\mathsf{t}}\, \mathsf{T} \quad . \tag{6.36}$$

This is a convenient result, because it means that if we ever need to express the normal coordinates in terms of the generalized displacements, we don't have to invert any matrices – we just need to do one matrix multiplication.

6.4.2 Summary of the method

(i) Obtain the T and V matrices,

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial T}{\partial q_\sigma\, \partial \dot q_{\sigma'}}\bigg|_{\bar q} \quad , \qquad \mathsf{V}_{\sigma\sigma'} = \frac{\partial U}{\partial q_\sigma\, \partial \dot q_{\sigma'}}\bigg|_{\bar q} \quad , \tag{6.37}$$

where the equilibrium conditions are $\partial U/\partial q_\sigma\big|_{\bar q} = 0$. The quadratic form Lagrangian for small oscillations of the generalized displacements from equilibrium η_σ and their velocities is then

$$L = \tfrac{1}{2}\, \dot\eta_\sigma\, \mathsf{T}_{\sigma\sigma'}\, \dot\eta_{\sigma'} - \tfrac{1}{2}\, \eta_\sigma\, \mathsf{V}_{\sigma\sigma'}\, \eta_{\sigma'} \quad . \tag{6.38}$$

(ii) Solve $\mathsf{det}(\omega^2\mathsf{T}-\mathsf{V})=0$, which is an $n^{\rm th}$ order polynomial in ω^2.

(iii) For each root ω_i^2, solve the defective linear system $(\omega_i^2\mathsf{T}-\mathsf{V})\,\boldsymbol{\psi}^{(i)}=0$.

(iv) Eigenvectors corresponding to different eigenfrequencies will necessarily be orthogonal, *i.e.*

$$\langle\,\boldsymbol{\psi}^{(i)}\,|\,\boldsymbol{\psi}^{(j)}\,\rangle\equiv\psi^{(i)}_\sigma\,\mathsf{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'}=0\quad\text{if}\quad\omega_i^2\neq\omega_j^2\quad. \tag{6.39}$$

In the case of degenerate eigenvalues, use the Gram-Schmidt method to find an orthogonal basis for the degenerate subspace. Then normalize each eigenvector such that $\langle\,\boldsymbol{\psi}^{(i)}\,|\,\mathsf{T}\,|\,\boldsymbol{\psi}^{(j)}\,\rangle=\delta_{ij}$ for all i and j.

(v) The modal matrix $\mathsf{A}_{\sigma j}=\psi^{(j)}_\sigma$ then satisfies

$$\mathsf{A}^{\mathsf{t}}\,\mathsf{T}\,\mathsf{A}=\mathbb{I}\qquad,\qquad\mathsf{A}^{\mathsf{t}}\,\mathsf{V}\,\mathsf{A}=\mathsf{diag}\big(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\big)\quad. \tag{6.40}$$

Note that $\mathsf{A}^{-1}=\mathsf{A}^{\mathsf{t}}\,\mathsf{T}$. The relation between the generalized displacements η_σ and the normal modes ξ_j is $\eta_\sigma=\mathsf{A}_{\sigma j}\,\xi_j$, which entails $\xi_j=\mathsf{A}^{\mathsf{t}}_{j\sigma}\mathsf{T}_{\sigma\sigma'}\eta_{\sigma'}=\mathsf{A}_{\sigma j}\mathsf{T}_{\sigma\sigma'}\eta_{\sigma'}$. In terms of the normal mode coordinates and their velocities,

$$L=\sum_i\tfrac{1}{2}\big(\dot\xi_i^2-\omega_i^2\,\xi_i^2\big)\quad, \tag{6.41}$$

and the equations of motion are those of decoupled oscillators: $\ddot\xi_i=-\omega_i^2\,\xi_i$.

(vi) The complete solution for the generalized displacements is then

$$\eta_\sigma(t)=\sum_{i=1}^n\mathsf{A}_{\sigma i}\,\Big\{C_i\cos(\omega_i t)+D_i\sin(\omega_i t)\Big\}\quad, \tag{6.42}$$

with

$$C_i=\mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\eta_{\sigma'}(0)\qquad,\qquad D_i=\omega_i^{-1}\mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\dot\eta_{\sigma'}(0)\quad. \tag{6.43}$$

6.5 Examples

6.5.1 Masses and springs

Two blocks and three springs are configured as in fig. 6.1. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.

(b) Find the T and V matrices.

(c) Suppose

$$m_1=2m\quad,\quad m_2=m\quad,\quad k_1=4k\quad,\quad k_2=k\quad,\quad k_3=2k\quad, \tag{6.44}$$

Find the frequencies of small oscillations.

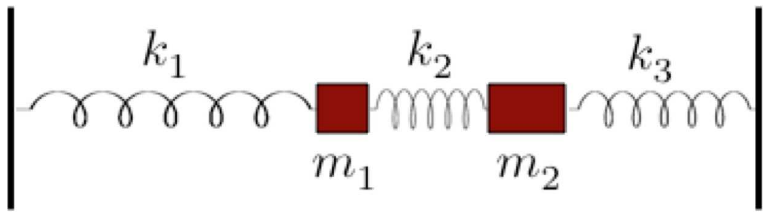

Figure 6.1: A system of masses and springs.

(d) Find the normal modes of oscillation.

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.

Solution :

(a) The Lagrangian is

$$L = \tfrac{1}{2}m_1\,\dot{x}_1^2 + \tfrac{1}{2}m_2\,\dot{x}_2^2 - \tfrac{1}{2}k_1\,x_1^2 - \tfrac{1}{2}k_2\,(x_2 - x_1)^2 - \tfrac{1}{2}k_3\,x_2^2 \quad , \tag{6.45}$$

which is already a quadratic form. Thus, the full equations of motion are already linear.

(b) The T and V matrices are

$$\mathsf{T}_{ij} = \frac{\partial^2 T}{\partial\dot{x}_i\,\partial\dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad , \qquad \mathsf{V}_{ij} = \frac{\partial^2 U}{\partial x_i\,\partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \quad . \tag{6.46}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda\,\omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2\mathsf{T} - \mathsf{V} = k\begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} \quad . \tag{6.47}$$

The determinant is

$$\begin{aligned} \det\,(\omega^2\mathsf{T} - \mathsf{V}) &= (2\lambda^2 - 11\lambda + 14)\,k^2 \\ &= (2\lambda - 7)\,(\lambda - 2)\,k^2 \quad . \end{aligned} \tag{6.48}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\omega_- = \sqrt{\frac{2k}{m}} \qquad , \qquad \omega_+ = \sqrt{\frac{7k}{2m}} \quad . \tag{6.49}$$

(d) The normal modes are determined from $(\omega_a^2 \mathsf{T} - \mathsf{V})\,\boldsymbol{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\boldsymbol{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boldsymbol{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad . \tag{6.50}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\boldsymbol{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boldsymbol{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad . \tag{6.51}$$

The standard normalization $\psi_i^{(a)}\, \mathsf{T}_{ij}\, \psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \qquad , \qquad \mathcal{C}_+ = \frac{1}{\sqrt{6m}} \quad . \tag{6.52}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) \quad . \tag{6.53}$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3} b \quad , \quad B = \tfrac{1}{3} b \quad , \quad C = 0 \quad , \quad D = 0 \quad . \tag{6.54}$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3} b \cdot \Big(2\cos(\omega_- t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3} b \cdot \Big(\cos(\omega_- t) - \cos(\omega_+ t) \Big) \quad . \end{aligned} \tag{6.55}$$

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega_- t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega_- t = \omega_+ t - \pi \quad \Rightarrow \quad t^* = \frac{2\pi}{\omega_- + \omega_+} \quad . \tag{6.56}$$

6.5.2 Double pendulum

As a second example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The Lagrangian and equations of motion for this problem were discussed in §4.4.5 for the general case of differing masses and lengths. For our simpler version, the kinetic and potential energies are

$$\begin{aligned} T &= m\ell^2 \dot{\theta}_1^2 + m\ell^2 \cos(\theta_1 - \theta_1)\, \dot{\theta}_1 \dot{\theta}_2 + \tfrac{1}{2} m\ell^2 \dot{\theta}_2^2 \\ U &= -2mg\ell \cos\theta_1 - mg\ell \cos\theta_2 \quad . \end{aligned} \tag{6.57}$$

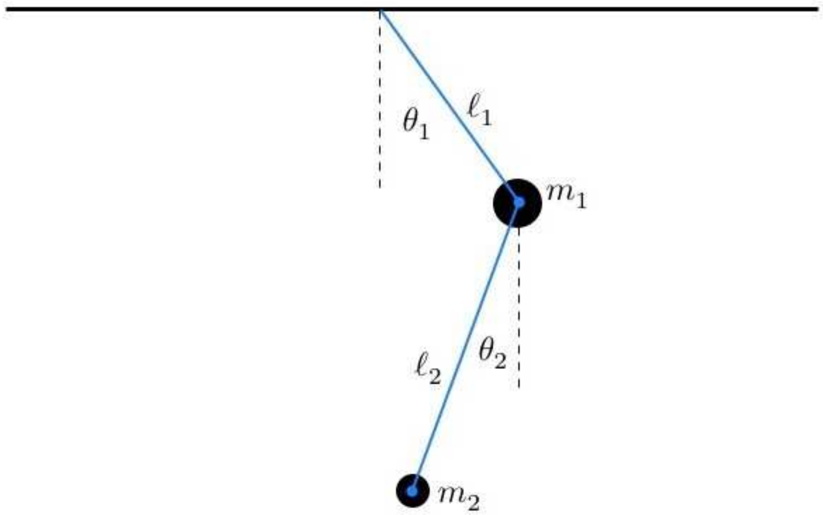

Figure 6.2: The double pendulum (again).

Equilibrium is at $\theta_1 = \theta_2 = 0$, and the T and V matrices are given by

$$\mathsf{T} = \frac{\partial^2 T}{\partial\dot\theta_i\,\partial\dot\theta_j}\bigg|_{\bar\theta} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \quad , \quad \mathsf{V} = \frac{\partial^2 U}{\partial\theta_i\,\partial\theta_j}\bigg|_{\bar\theta} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} \quad . \tag{6.58}$$

Then

$$\omega^2\mathsf{T} - \mathsf{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} \quad , \tag{6.59}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2 \quad . \tag{6.60}$$

We find the unnormalized eigenvectors by setting $(\omega_i^2\,\mathsf{T} - V)\,\psi^{(i)} = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad , \quad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \quad , \tag{6.61}$$

where $C_\pm$ are constants. One can check $\mathsf{T}_{\sigma\sigma'}\,\psi_\sigma^{(i)}\,\psi_{\sigma'}^{(j)}$ vanishes for $i \neq j$. We then normalize by demanding $\mathsf{T}_{\sigma\sigma'}\,\psi_\sigma^{(i)}\,\psi_{\sigma'}^{(i)} = 1$ (no sum on i), which determines the coefficients $C_\pm = \frac{1}{2}\sqrt{(2 \pm \sqrt{2})/m\ell^2}$. Thus, the modal matrix is

$$\mathsf{A} = \begin{pmatrix} \psi_1^+ & \psi_1^- \\ \psi_2^+ & \psi_2^- \end{pmatrix} = \frac{1}{2\sqrt{m\ell^2}} \begin{pmatrix} \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{4+2\sqrt{2}} & +\sqrt{4-2\sqrt{2}} \end{pmatrix} \quad . \tag{6.62}$$

6.6 Zero Modes

6.6.1 Noether's theorem and zero modes

Recall Noether's theorem, which says that for every continuous one-parameter family of coordinate transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q,\zeta) \qquad , \qquad \tilde{q}_\sigma(q,\zeta=0) = q_\sigma \quad , \tag{6.63}$$

which leaves the Lagrangian invariant, *i.e.* $dL/d\zeta = 0$, there is an associated conserved quantity,

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \qquad \text{satisfies} \qquad \frac{d\Lambda}{dt} = 0 \quad . \tag{6.64}$$

For small oscillations, we write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$, hence

$$\Lambda_k = \sum_\sigma C_{k\sigma}\, \dot{\eta}_\sigma \quad , \tag{6.65}$$

where k labels the one-parameter families (in the event there is more than one continuous symmetry), and where

$$C_{k\sigma} = \sum_{\sigma'} \mathsf{T}_{\sigma\sigma'}\, \frac{\partial \tilde{q}_{\sigma'}}{\partial \zeta_k}\bigg|_{\zeta=0} \quad . \tag{6.66}$$

Therefore, we can define the (unnormalized) normal mode

$$\xi_k = \sum_\sigma C_{k\sigma}\, \eta_\sigma \quad , \tag{6.67}$$

which satisfies $\ddot{\xi}_k = 0$. Thus, in systems with continuous symmetries, to each such continuous symmetry there is an associated zero mode of the small oscillations problem, *i.e.* a mode with $\omega_k^2 = 0$.

6.6.2 Examples of zero modes

The simplest example of a zero mode would be a pair of masses m_1 and m_2 moving frictionlessly along a line and connected by a spring of force constant k and unstretched length a. We know from our study of central forces that the Lagrangian may be written

$$\begin{aligned} L &= \tfrac{1}{2} m_1\, \dot{x}_1^2 + \tfrac{1}{2} m_2\, \dot{x}_2^2 - \tfrac{1}{2} k\,(x_1 - x_2 - a)^2 \\ &= \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2}\mu \dot{x}^2 - \tfrac{1}{2} k (x-a)^2 \quad , \end{aligned} \tag{6.68}$$

where $X = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$ is the center of mass position, $x = x_1 - x_2$ is the relative coordinate, $M = m_1 + m_2$ is the total mass, and $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. The relative coordinate obeys $\ddot{x} = -\omega_0^2\, x$, where the oscillation frequency is $\omega_0 = \sqrt{k/\mu}$. The center of mass coordinate obeys $\ddot{X} = 0$, *i.e.* its oscillation frequency is zero. The center of mass motion is a zero mode.

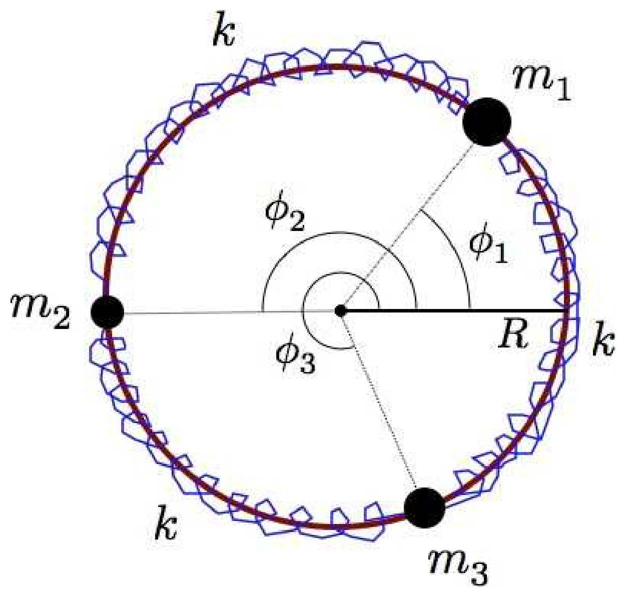

Figure 6.3: Coupled oscillations of three masses on a frictionless hoop of radius R. All three springs have the same force constant k, but the masses are all distinct.

Another example is furnished by the system depicted in fig. 6.3, where three distinct masses m_1, m_2, and m_3 move around a frictionless hoop of radius R. The masses are connected to their neighbors by identical springs of force constant k. We choose as generalized coordinates the angles ϕ_σ ($\sigma = 1, 2, 3$), with the convention that

$$\phi_3 - 2\pi < \phi_1 \le \phi_2 \le \phi_3 \le 2\pi + \phi_1 \quad . \tag{6.69}$$

The kinetic energy is

$$T = \tfrac{1}{2}R^2\big(m_1\,\dot\phi_1^2 + m_2\,\dot\phi_2^2 + m_3\,\dot\phi_3^3\big) \quad . \tag{6.70}$$

Let $R\chi$ be the equilibrium length for each of the springs. Then the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}kR^2\,\Big\{(\phi_2-\phi_1-\chi)^2 + (\phi_3-\phi_2-\chi)^2 + (2\pi+\phi_1-\phi_3-\chi)^2\Big\} \\ &= \tfrac{1}{2}kR^2\,\Big\{(\phi_2-\phi_1)^2 + (\phi_3-\phi_2)^2 + (2\pi+\phi_1-\phi_3)^2 + 3\chi^2 - 4\pi\chi\Big\} \quad . \end{aligned} \tag{6.71}$$

Note that the equilibrium angle χ enters only in an additive constant to the potential energy. Thus, for the calculation of the equations of motion, it is irrelevant. It doesn't matter whether or not the equilibrium configuration is unstretched ($\chi = 2\pi/3$) or not ($\chi \ne 2\pi/3$).

The equilibrium configuration is

$$\bar\phi_1 = \zeta \quad , \quad \bar\phi_1 = \zeta + \tfrac{2\pi}{3} \quad , \quad \bar\phi_1 = \zeta + \tfrac{4\pi}{3} \quad , \tag{6.72}$$

where ζ is an arbitrary real number, corresponding to continuous translational invariance of the entire system around the ring. The T and V matrices are then

$$\mathsf{T} = \begin{pmatrix} m_1R^2 & 0 & 0 \\ 0 & m_2R^2 & 0 \\ 0 & 0 & m_3R^2 \end{pmatrix} \quad , \qquad \mathsf{V} = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix} \quad . \tag{6.73}$$

We then have

$$\omega^2\,\mathsf{T} - \mathsf{V} = kR^2 \begin{pmatrix} \frac{\omega^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega^2}{\Omega_3^2} - 2 \end{pmatrix} \quad , \tag{6.74}$$

where $\Omega_j^2 \equiv k/m_j$. We compute the determinant to find the characteristic polynomial:

$$\begin{aligned} P(\omega) &= \det(\omega^2\,\mathsf{T} - \mathsf{V}) \equiv (kR^2)^3\,\widetilde{P}(\omega) \\ \widetilde{P}(\omega) &= \frac{\omega^6}{\Omega_1^2\,\Omega_2^2\,\Omega_3^2} - 2\left(\frac{1}{\Omega_1^2\,\Omega_2^2} + \frac{1}{\Omega_2^2\,\Omega_3^2} + \frac{1}{\Omega_1^2\,\Omega_3^2}\right)\omega^4 + 3\left(\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \frac{1}{\Omega_3^2}\right)\omega^2 \quad . \end{aligned} \tag{6.75}$$

The equation $\widetilde{P}(\omega) = 0$ yields a cubic equation in ω^2, but clearly ω^2 is a factor, and when we divide this out we obtain a quadratic equation. One root obviously is $\omega_1^2 = 0$. The other two roots are solutions to the quadratic equation:

$$\omega_{2,3}^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 \pm \sqrt{\tfrac{1}{2}\big(\Omega_1^2 - \Omega_2^2\big)^2 + \tfrac{1}{2}\big(\Omega_2^2 - \Omega_3^2\big)^2 + \tfrac{1}{2}\big(\Omega_1^2 - \Omega_3^2\big)^2} \quad . \tag{6.76}$$

To find the eigenvectors and the modal matrix, we set

$$\begin{pmatrix} \frac{\omega_j^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega_j^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega_j^2}{\Omega_3^2} - 2 \end{pmatrix} \begin{pmatrix} \psi_1^{(j)} \\ \psi_2^{(j)} \\ \psi_3^{(j)} \end{pmatrix} = 0 \quad , \tag{6.77}$$

Writing down the three coupled equations for the components of $\boldsymbol{\psi}^{(j)}$, we find

$$\left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)\psi_1^{(j)} = \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)\psi_2^{(j)} = \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)\psi_3^{(j)} \quad . \tag{6.78}$$

We therefore conclude

$$\boldsymbol{\psi}^{(j)} = \mathcal{C}_j \begin{pmatrix} \left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)^{-1} \end{pmatrix} \quad . \tag{6.79}$$

The normalization condition $\psi_\sigma^{(i)}\,\mathsf{T}_{\sigma\sigma'}\,\psi_{\sigma'}^{(j)} = \delta_{ij}$ then fixes the constants $\mathcal{C}_j$:

$$\left[m_1\left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)^{-2} + \; m_2\left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)^{-2} + \; m_3\left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)^{-2}\right]\big|\mathcal{C}_j\big|^2 = 1 \quad . \tag{6.80}$$

The Lagrangian is invariant under the one-parameter family of transformations

$$\phi_\sigma \longrightarrow \phi_\sigma + \zeta \tag{6.81}$$

for all $\sigma = 1, 2, 3$. The associated conserved quantity is

$$\begin{aligned}\Lambda &= \sum_\sigma \frac{\partial L}{\partial \dot{\phi}_\sigma} \frac{\partial \tilde{\phi}_\sigma}{\partial \zeta} \\ &= R^2 \left(m_1 \, \dot{\phi}_1 + m_2 \, \dot{\phi}_2 + m_3 \, \dot{\phi}_3 \right) \quad ,\end{aligned} \tag{6.82}$$

which is, of course, the total angular momentum relative to the center of the ring. We stress that Λ is a constant in general, and not only in the limit of small deviations from static equilibrium. From

$$\xi_1 = \mathcal{C} \left(m_1 \, \eta_1 + m_2 \, \eta_2 + m_3 \, \eta_3 \right) \quad , \tag{6.83}$$

where $\mathcal{C}$ is a constant. Recall the relation $\eta_\sigma = \mathsf{A}_{\sigma i} \, \xi_i$ between the generalized displacements η_σ and the normal coordinates ξ_i, which may be inverted to yield $\xi_i = \mathsf{A}^{-1}_{i\sigma} \, \eta_\sigma = \mathsf{A}^{\mathsf{t}}_{i\sigma}$. In our case here, the T matrix is diagonal, so the multiplication is trivial. From eqns. 6.83 and 6.35, we conclude that the matrix $\mathsf{A}^{\mathsf{t}}\,\mathsf{T}$ must have a first *row* which is proportional to (m_1, m_2, m_3). Since these are the very diagonal entries of T, we conclude that A^{t} itself must have a first row which is proportional to $(1, 1, 1)$, which means that the first *column* of A is proportional to $(1, 1, 1)$. But this is confirmed by eqn. 6.78 when we take $j = 1$, since $\omega^2_{j=1} = 0$: $\psi^{(1)}_1 = \psi^{(1)}_2 = \psi^{(1)}_3$.

6.7 Chain of Mass Points

6.7.1 Lagrangian and equations of motion

Next consider an infinite chain of identical masses, connected by identical springs of spring constant k and equilibrium length a. The Lagrangian is

$$\begin{aligned}L &= \tfrac{1}{2} m \sum_n \dot{x}^2_n - \tfrac{1}{2} k \sum_n (x_{n+1} - x_n - a)^2 \\ &= \tfrac{1}{2} m \sum_n \dot{u}^2_n - \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \quad ,\end{aligned} \tag{6.84}$$

where $u_n \equiv x_n - na + \zeta$ is the displacement from equilibrium of the n^{th} mass. The constant ζ is arbitrary and is cyclic in L, reflecting overall translational invariance with a consequent zero mode according to Noether's theorem. The Euler-Lagrange equations are

$$\begin{aligned}m\ddot{u}_n &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_n}\right) = \frac{\partial L}{\partial u_n} \\ &= k\,(u_{n+1} - u_n) - k\,(u_n - u_{n-1}) = k\,(u_{n+1} + u_{n-1} - 2u_n) \quad .\end{aligned} \tag{6.85}$$

Now let us assume that the system is placed on a large ring of circumference Na, where $N \gg 1$. Then $u_{n+N} = u_n$ and we may shift to Fourier coefficients,

$$u_n = \frac{1}{\sqrt{N}} \sum_q e^{iqan} \, \hat{u}_q \qquad , \qquad \hat{u}_q = \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \, u_n \quad , \tag{6.86}$$

where $q_j = 2\pi j/Na$, and both sums are over the set $j, n \in \{1, \ldots, N\}$. Expressed in terms of the $\{\hat{u}_q\}$, the equations of motion become

$$\begin{aligned}\ddot{\hat{u}}_q &= \frac{1}{\sqrt{N}} \sum_n e^{-iqna}\, \ddot{u}_n = \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \left(u_{n+1} + u_{n-1} - 2u_n\right) \\ &= \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \left(e^{-iqa} + e^{+iqa} - 2\right) u_n = -\frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \hat{u}_q\end{aligned} \tag{6.87}$$

Thus, the $\{\hat{u}_q\}$ *are* the normal modes of the system (up to a normalization constant), and the eigenfrequencies are

$$\omega_q = 2\sqrt{\frac{k}{m}}\, \left|\sin\left(\tfrac{1}{2}qa\right)\right| \quad . \tag{6.88}$$

This means that the modal matrix is

$$\mathsf{A}_{nq} = \frac{1}{\sqrt{Nm}}\, e^{iqan} \quad , \tag{6.89}$$

where we've included the $\frac{1}{\sqrt{m}}$ factor for a proper normalization. The normal modes themselves are then $\xi_q = \mathsf{A}^\dagger_{qn}\, \mathsf{T}_{nn'}\, u_{n'} = \sqrt{m}\, \hat{u}_q$. For complex A, we have $\mathsf{A}^\dagger \mathsf{T} \mathsf{A} = \mathbb{I}$ and $\mathsf{A}^\dagger \mathsf{V} \mathsf{A} = \mathsf{diag}(\omega_1^2, \ldots, \omega_N^2)$.

Note that

$$\begin{aligned}\mathsf{T}_{nn'} &= m\, \delta_{n,n'} \\ \mathsf{V}_{nn'} &= 2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1}\end{aligned} \tag{6.90}$$

and that

$$\begin{aligned}(\mathsf{A}^\dagger \mathsf{T} \mathsf{A})_{qq'} &= \sum_{n=1}^{N} \sum_{n'=1}^{N} \mathsf{A}^*_{nq} \mathsf{T}_{nn'} \mathsf{A}_{n'q'} \\ &= \frac{1}{Nm} \sum_{n=1}^{N} \sum_{n'=1}^{N} e^{-iqan}\, m\, \delta_{nn'}\, e^{iq'an'} = \frac{1}{N} \sum_{n=1}^{N} e^{i(q'-q)an} = \delta_{qq'} \quad ,\end{aligned} \tag{6.91}$$

and

$$\begin{aligned}(\mathsf{A}^\dagger \mathsf{V} \mathsf{A})_{qq'} &= \sum_{n=1}^{N} \sum_{n'=1}^{N} \mathsf{A}^*_{nq} \mathsf{T}_{nn'} \mathsf{A}_{n'q'} \\ &= \frac{1}{Nm} \sum_{n=1}^{N} \sum_{n'=1}^{N} e^{-iqan} \Big(2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1}\Big)\, e^{iq'an'} \\ &= \frac{k}{m} \frac{1}{N} \sum_{n=1}^{N} e^{i(q'-q)an} \left(2 - e^{-iq'a} - e^{iq'a}\right) = \frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \delta_{qq'} = \omega_q^2\, \delta_{qq'}\end{aligned} \tag{6.92}$$

Since $\hat{x}_{q+G} = \hat{x}_q$, where $G = 2\pi/a$, we may choose any set of q values such that no two are separated by an integer multiple of G. The set of points $\{jG\}$ with $j \in \mathbb{Z}$ is called the *reciprocal lattice*. For a linear chain, the reciprocal lattice is itself a linear chain[2]. One natural set to choose is $q \in \big[-\frac{\pi}{a}, \frac{\pi}{a}\big]$. This is known as the *first Brillouin zone* of the reciprocal lattice.

Finally, we can write the Lagrangian itself in terms of the $\{u_q\}$. One easily finds

$$L = \tfrac{1}{2}\, m \sum_q \dot{\hat{u}}_q^* \, \dot{\hat{u}}_q - \, k \sum_q (1 - \cos qa)\, \hat{u}_q^* \, \hat{u}_q \quad , \tag{6.93}$$

where the sum is over q in the first Brillouin zone. Note that

$$\hat{u}_{-q} = \hat{u}_{-q+G} = \hat{u}_q^* \quad . \tag{6.94}$$

This means that we can restrict the sum to half the Brillouin zone:

$$L = \sum_{q \in [0, \frac{\pi}{a}]} \left\{ m \, \dot{\hat{u}}_q^* \, \dot{\hat{u}}_q - 4k \, \sin^2\!\big(\tfrac{1}{2} qa\big)\, \hat{u}_q^* \, \hat{u}_q \right\} \quad . \tag{6.95}$$

Now $\hat{u}_q$ and $\hat{u}_q^*$ may be regarded as linearly independent, as one regards complex variables z and z^*. The Euler-Lagrange equation for $\hat{u}_q^*$ gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\hat{u}}_q^*}\right) = \frac{\partial L}{\partial \hat{u}_q^*} \quad \Rightarrow \quad \ddot{\hat{u}}_q = -\omega_q^2 \, \hat{u}_q \quad . \tag{6.96}$$

Extremizing with respect to $\hat{u}_q$ gives the complex conjugate equation.

6.7.2 Continuum limit

Let us take $N \to \infty$, $a \to 0$, with $L_0 = Na$ fixed. We'll write $u_n(t) \to u(x = na, t)$, in which case

$$\begin{aligned} T &= \tfrac{1}{2} m \sum_n \dot{u}_n^2 \quad \longrightarrow \quad \tfrac{1}{2} m \int \frac{dx}{a} \left(\frac{\partial u}{\partial t}\right)^2 \\ V &= \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \quad \longrightarrow \quad \tfrac{1}{2} k \int \frac{dx}{a} \left(\frac{u(x+a) - u(x)}{a}\right)^2 a^2 \end{aligned} \tag{6.97}$$

Recognizing the spatial derivative above, we finally obtain

$$\begin{aligned} L &= \int \! dx \, \mathcal{L}(u, \partial_t u, \partial_x u) \\ \mathcal{L} &= \tfrac{1}{2}\, \mu \left(\frac{\partial u}{\partial t}\right)^2 - \tfrac{1}{2}\, \tau \left(\frac{\partial u}{\partial x}\right)^2 \quad , \end{aligned} \tag{6.98}$$

where $\mu = m/a$ is the linear mass density and $\tau = ka$ is the tension[3]. The quantity $\mathcal{L}$ is the *Lagrangian density*; it depends on the field $u(x,t)$ as well as its partial derivatives $\partial_t u$ and $\partial_x u$[4]. The action is

$$S\big[u(x,t)\big] = \int\limits_{t_a}^{t_b} \! dt \int\limits_{x_a}^{x_b} \! dx \, \mathcal{L}(u, \partial_t u, \partial_x u) \quad , \tag{6.99}$$

[2]For higher dimensional Bravais lattices, the reciprocal lattice is often different than the real space ("direct") lattice. For example, the reciprocal lattice of a face-centered cubic structure is a body-centered cubic lattice.

[3]For a proper limit, we demand μ and τ be neither infinite nor infinitesimal.

[4]$\mathcal{L}$ may also depend explicitly on x and t.

where $\{x_a, x_b\}$ are the limits on the x coordinate. Setting $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t u\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_x u\right)}\right) = 0 \quad . \tag{6.100}$$

For our system, this yields the Helmholtz equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad , \tag{6.101}$$

where $c = \sqrt{\tau/\mu}$ is the velocity of wave propagation. This is a linear equation, solutions of which are of the form

$$u(x,t) = C\, e^{iqx}\, e^{-i\omega t} \quad , \tag{6.102}$$

where $\omega = \pm cq$. Note that in the continuum limit $a \to 0$, the dispersion relation derived for the chain becomes

$$\omega_q^2 = \frac{4k}{m}\sin^2\left(\tfrac{1}{2}qa\right) \longrightarrow \frac{ka^2}{m}\, q^2 = c^2 q^2 \quad , \tag{6.103}$$

and so the results agree.

6.8 General Formulation of Small Oscillations

In the development in section 6.2, we assumed that the kinetic energy T is a homogeneous function of degree 2, and the potential energy U a homogeneous function of degree 0, in the generalized velocities $\dot{q}_\sigma$. However, we've encountered situations where this is not so: problems with time-dependent holonomic constraints, such as the mass point on a rotating hoop, and problems involving charged particles moving in magnetic fields. The general Lagrangian is of the form

$$L = \tfrac{1}{2}\, T_2^{\sigma\sigma'}(q)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} + T_1^{\sigma}(q)\, \dot{q}_\sigma + T_0(q) - U_1^{\sigma}(q)\, \dot{q}_\sigma - U_0(q) \quad , \tag{6.104}$$

where the subscript 0, 1, or 2 labels the degree of homogeneity of each term in the generalized velocities. The generalized momenta are then

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_2^{\sigma\sigma'}\, \dot{q}_{\sigma'} + T_1^{\sigma}(q) - U_1^{\sigma}(q) \tag{6.105}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{\partial (T_0 - U_0)}{\partial q_\sigma} + \frac{\partial (T_1^{\sigma'} - U_1^{\sigma'})}{\partial q_\sigma}\, \dot{q}_{\sigma'} + \frac{1}{2}\, \frac{\partial T_2^{\sigma\sigma'}}{\partial q_\sigma}\, \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} \quad , \tag{6.106}$$

and the equations of motion are again $\dot{p}_\sigma = F_\sigma$

In equilibrium, we seek a time-independent solution of the form $q_\sigma(t) = \bar{q}_\sigma$. This entails

$$\frac{\partial\left\{U_0(q) - T_0(q)\right\}}{\partial q_\sigma}\Bigg|_{q=\bar{q}} = 0 \quad , \tag{6.107}$$

which give us n equations in the n unknowns $(\bar{q}_1, \ldots, \bar{q}_n)$. We then write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$ and expand in the notionally small quantities η_σ. It is important to understand that we assume η and all of its time derivatives as well are small. Thus, we can expand L to quadratic order in $(\eta, \dot{\eta})$ to obtain

$$L = \tfrac{1}{2}\,\mathsf{T}_{\sigma\sigma'}\,\dot{\eta}_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\,\mathsf{B}_{\sigma\sigma'}\,\eta_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\,\mathsf{V}_{\sigma\sigma'}\,\eta_\sigma\,\eta_{\sigma'} \quad , \tag{6.108}$$

where

$$\mathsf{T}_{\sigma\sigma'} = T_2^{\sigma\sigma'}(\bar{q}) \quad , \quad \mathsf{B}_{\sigma\sigma'} = 2\,\frac{\partial\big(U_1^{\sigma'} - T_1^{\sigma'}\big)}{\partial q_\sigma}\bigg|_{q=\bar{q}} \quad , \quad \mathsf{V}_{\sigma\sigma'} = \frac{\partial^2\big(U_0 - T_0\big)}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q=\bar{q}} \quad . \tag{6.109}$$

Note that the T and V matrices are symmetric. The $\mathsf{B}_{\sigma\sigma'}$ term is new.

Now we can always write $\mathsf{B} = \frac{1}{2}(\mathsf{B}^{\mathsf{s}} + \mathsf{B}^{\mathsf{a}})$ as a sum over symmetric and antisymmetric parts, with $\mathsf{B}^{\mathsf{s}} = \frac{1}{2}(\mathsf{B} + \mathsf{B}^{\mathsf{t}})$ and $\mathsf{B}^{\mathsf{a}} = \frac{1}{2}(\mathsf{B} - \mathsf{B}^{\mathsf{t}})$. Since,

$$\mathsf{B}_{\sigma\sigma'}\,\eta_\sigma\,\dot{\eta}_{\sigma'} = \frac{d}{dt}\Big(\tfrac{1}{2}\,\mathsf{B}^{\mathsf{s}}_{\sigma\sigma'}\,\eta_\sigma\,\eta_{\sigma'}\Big) \quad , \tag{6.110}$$

any symmetric part to B contributes a total time derivative to L, and thus has no effect on the equations of motion. Therefore, we can project V onto its antisymmetric part, writing

$$\mathsf{B}_{\sigma\sigma'} \equiv \mathsf{B}^{\mathsf{A}}_{\sigma\sigma'} = \left(\frac{\partial\big(U_1^{\sigma'} - T_1^{\sigma'}\big)}{\partial q_\sigma} - \frac{\partial\big(U_1^{\sigma} - T_1^{\sigma}\big)}{\partial q_{\sigma'}}\right)_{q=\bar{q}} \quad . \tag{6.111}$$

We now have

$$p_\sigma = \frac{\partial L}{\partial \dot{\eta}_\sigma} = \mathsf{T}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \tfrac{1}{2}\,\mathsf{B}_{\sigma\sigma'}\,\eta_{\sigma'} \quad , \tag{6.112}$$

and

$$F_\sigma = \frac{\partial L}{\partial \eta_\sigma} = -\tfrac{1}{2}\,\mathsf{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} - \mathsf{V}_{\sigma\sigma'}\,\eta_{\sigma'} \quad . \tag{6.113}$$

The equations of motion, $\dot{p}_\sigma = F_\sigma$, then yield

$$\mathsf{T}_{\sigma\sigma'}\,\ddot{\eta}_{\sigma'} + \mathsf{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \mathsf{V}_{\sigma\sigma'}\,\eta_{\sigma'} = 0 \quad . \tag{6.114}$$

Let us write $\boldsymbol{\eta}(t) = \boldsymbol{\eta}\,e^{-i\omega t}$. We then have

$$\big(\omega^2\,\mathsf{T} + i\omega\,\mathsf{B} - \mathsf{V}\big)\boldsymbol{\eta} = 0 \quad . \tag{6.115}$$

To solve eqn. 6.115, we set $P(\omega) = 0$, where $P(\omega) = \det\big[\mathsf{Q}(\omega)\big]$, with

$$\mathsf{Q}(\omega) \equiv \omega^2\,\mathsf{T} + i\omega\,\mathsf{B} - \mathsf{V} \quad . \tag{6.116}$$

Since T, B, and V are real-valued matrices, and since $\det(M) = \det(M^{\mathsf{t}})$ for any matrix M, we can use $\mathsf{B}^{\mathsf{t}} = -\mathsf{B}$ to obtain $P(-\omega) = P(\omega)$ and $P(\omega^*) = \big[P(\omega)\big]^*$. This establishes that if $P(\omega) = 0$, *i.e.* if ω is an eigenfrequency, then $P(-\omega) = 0$ and $P(\omega^*) = 0$, *i.e.* $-\omega$ and ω^* and $-\omega^*$ are also eigenfrequencies. Furthermore, $P(\omega)$ must again be a polynomial of order n in ω^2.

Example

As an example, consider the following Lagrangian, which is a function of four generalized coordinates $\{x_1, y_1, x_2, y_2\}$ and their corresponding velocities:

$$L = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\big) - \tfrac{1}{2}\kappa\big(x_1 - x_2\big)^2 - \tfrac{1}{2}b\big(y_1^2 + y_2^2\big) + \tfrac{1}{2}m\omega_{\rm c}\big(x_1\,\dot{y}_1 - y_1\,\dot{x}_1 + x_2\,\dot{y}_2 - y_2\,\dot{x}_2\big) \quad . \tag{6.117}$$

The last term, which is linear in the generalized velocities, arises if the masses are also equally charged and in the presence of a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The quantity $\omega_{\rm c} = qB/mc$, where q is the charge, is called the *cyclotron frequency*. We then have

$$\begin{aligned}
p_{x,1} &= m\dot{x}_1 - \tfrac{1}{2}m\omega_{\rm c}\,y_1 \quad , \qquad & F_{x,1} &= -\kappa\big(x_1 - x_2\big) + \tfrac{1}{2}m\omega_{\rm c}\,\dot{y}_1 \\
p_{y,1} &= m\dot{y}_1 + \tfrac{1}{2}m\omega_{\rm c}\,x_1 \quad , \qquad & F_{y,1} &= -by_1 - \tfrac{1}{2}m\omega_{\rm c}\,\dot{x}_1 \\
p_{x,2} &= m\dot{x}_2 - \tfrac{1}{2}m\omega_{\rm c}\,y_2 \quad , \qquad & F_{x,2} &= -\kappa\big(x_2 - x_1\big) + \tfrac{1}{2}m\omega_{\rm c}\,\dot{y}_2 \\
p_{y,2} &= m\dot{y}_2 + \tfrac{1}{2}m\omega_{\rm c}\,x_2 \quad , \qquad & F_{y,2} &= -by_2 - \tfrac{1}{2}m\omega_{\rm c}\,\dot{x}_2 \quad .
\end{aligned} \tag{6.118}$$

Defining $\nu^2 \equiv \kappa/m$ and $\Omega^2 \equiv b/m$, we have the equations of motion

$$\begin{aligned}
\ddot{x}_1 - \omega_{\rm c}\,\dot{y}_1 &= -\nu^2\big(x_1 - x_2\big) \\
\ddot{y}_1 + \omega_{\rm c}\,\dot{x}_1 &= -\Omega^2\,y_1 \\
\ddot{x}_2 - \omega_{\rm c}\,\dot{y}_2 &= -\nu^2\big(x_2 - x_1\big) \\
\ddot{y}_2 + \omega_{\rm c}\,\dot{x}_2 &= -\Omega^2\,y_2 \quad .
\end{aligned} \tag{6.119}$$

From these equations, we read off the matrices

$$\mathsf{T} = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \quad , \quad \mathsf{B} = \begin{pmatrix} 0 & -m\omega_{\rm c} & 0 & 0 \\ m\omega_{\rm c} & 0 & 0 & 0 \\ 0 & 0 & 0 & m\omega_{\rm c} \\ 0 & 0 & -m\omega_{\rm c} & 0 \end{pmatrix} \tag{6.120}$$

and

$$\mathsf{V} = \begin{pmatrix} m\nu^2 & 0 & -m\nu^2 & 0 \\ 0 & m\Omega^2 & 0 & 0 \\ -m\nu^2 & 0 & m\nu^2 & 0 \\ 0 & 0 & 0 & m\Omega^2 \end{pmatrix} \quad , \tag{6.121}$$

where the rows and columns correspond to the coordinates $\{x_1, y_1, x_2, y_2\}$, respectively. If we define the CM and relative coordinates

$$X \equiv \tfrac{1}{2}(x_1 + x_2) \quad , \quad Y \equiv \tfrac{1}{2}(y_1 + y_2) \quad , \quad x \equiv x_1 - x_2 \quad , \quad y \equiv y_1 - y_2 \quad , \tag{6.122}$$

the equations of motion decouple into two 2×2 systems, *viz.*

$$\ddot{X} - \omega_{\rm c}\,\dot{Y} = 0 \quad , \quad \ddot{Y} + \omega_{\rm c}\,\dot{X} = -\Omega^2\,Y \quad , \quad \ddot{x} - \omega_{\rm c}\,\dot{y} = -2\nu^2 x \quad , \quad \ddot{y} + \omega_{\rm c}\,\dot{x} = -\Omega^2\,y \quad . \tag{6.123}$$

Thus, for the (X,Y) system we have

$$\det\begin{pmatrix} \omega^2 & -i\omega\omega_\mathrm{c} \\ i\omega\omega_\mathrm{c} & \omega^2-\varOmega^2 \end{pmatrix} = 0 \qquad \Rightarrow \qquad \omega_1^2 = 0 \quad , \quad \omega_2^2 = \varOmega^2 + \omega_\mathrm{c}^2 \quad , \tag{6.124}$$

while for the (x,y) system we have

$$\det\begin{pmatrix} \omega^2-2\nu^2 & -i\omega\omega_\mathrm{c} \\ i\omega\omega_\mathrm{c} & \omega^2-\varOmega^2 \end{pmatrix} = 0 \qquad \Rightarrow \qquad \omega_{3,4}^2 = \tfrac{1}{2}\big(2\nu^2+\varOmega^2+\omega_\mathrm{c}^2\big) \pm \tfrac{1}{2}\sqrt{\big(2\nu^2+\varOmega^2+\omega_\mathrm{c}^2\big)^2 - 8\nu^2\varOmega^2} \quad . \tag{6.125}$$

When $\omega_\mathrm{c}=0$, we have the zero mode X with frequency $\omega_1=0$, the relative coordinate x with frequency $\omega_4=\sqrt{2}\nu$, and two independent y and Y oscillations with degenerate frequencies $\omega_2=\omega_3=\varOmega$. Nonzero ω_c couples x to y and X to Y, and shifts the eigenfrequencies $\omega_{2,3,4}$ according to the above results.

Note that zero mode frequency is unaffected by a finite ω_c. If we write the Lagrangian in terms of the CM and relative coordinates, we obtain $L=L_\mathrm{CM}+L_\mathrm{rel}$, with

$$\begin{aligned} L_\mathrm{CM} &= m\big(\dot{X}^2+\dot{Y}^2\big) + m\omega_\mathrm{c}\big(X\dot{Y}-Y\dot{X}\big) - bY^2 \\ L_\mathrm{rel} &= \tfrac{1}{4}m\big(\dot{x}^2+\dot{y}^2\big) + \tfrac{1}{4}m\omega_\mathrm{c}\big(x\dot{y}-y\dot{x}\big) - \tfrac{1}{2}\kappa x^2 - \tfrac{1}{4}by^2 \quad . \end{aligned} \tag{6.126}$$

At first, it seems that the zero mode should be lifted by finite ω_c since the coordinate X is no longer cyclic in L. However, X may be made cyclic by a different *choice of gauge* for the electromagnetic vector potential. Our choice had been $\boldsymbol{A}(\boldsymbol{r})=\frac{1}{2}B\hat{\boldsymbol{z}}\times\boldsymbol{r}=\frac{1}{2}B(x\hat{\boldsymbol{y}}-y\hat{\boldsymbol{x}})$, but had we instead chosen $\boldsymbol{A}=-By\hat{\boldsymbol{x}}$, we would have had $\frac{q}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{r}}=-\frac{qB}{c}\,y\,\dot{x}$ and only the velocities $\dot{x}_{1,2}$ would have entered here for each particle, so X would have been cyclic. Equivalently, in L_CM we could write

$$m\omega_\mathrm{c}\big(X\dot{Y}-Y\dot{X}\big) = \frac{d}{dt}\big(m\omega_\mathrm{c}\,XY\big) - 2m\omega_\mathrm{c}Y\dot{X} \quad , \tag{6.127}$$

and the total time derivative term may be dropped from L_CM. The resulting CM Lagrangian is then cyclic in X, so the zero mode survives!

6.9 Additional Examples

6.9.1 Right triatomic molecule

A molecule consists of three identical atoms located at the vertices of a $45°$ right triangle. Each pair of atoms interacts by an effective spring potential, with all spring constants equal to k. Consider only planar motion of this molecule.

(a) Find three 'zero modes' for this system (*i.e.* normal modes whose associated eigenfrequencies vanish).

(b) Find the remaining three normal modes.

Solution

It is useful to choose the following coordinates:

$$\begin{aligned}(X_1, Y_1) &= (x_1\ ,\ y_1)\\ (X_2, Y_2) &= (a + x_2\ ,\ y_2)\\ (X_3, Y_3) &= (x_3\ ,\ a + y_3)\quad .\end{aligned}\tag{6.128}$$

The three separations are then

$$\begin{aligned}d_{12} &= \sqrt{(a + x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= a + x_2 - x_1 + \ldots\end{aligned}\tag{6.129}$$

and

$$\begin{aligned}d_{23} &= \sqrt{(-a + x_3 - x_2)^2 + (a + y_3 - y_2)^2}\\ &= \sqrt{2}\,a - \tfrac{1}{\sqrt{2}}\big(x_3 - x_2\big) + \tfrac{1}{\sqrt{2}}\big(y_3 - y_2\big) + \ldots\end{aligned}\tag{6.130}$$

and

$$\begin{aligned}d_{13} &= \sqrt{(x_3 - x_1)^2 + (a + y_3 - y_1)^2}\\ &= a + y_3 - y_1 + \ldots\quad .\end{aligned}\tag{6.131}$$

The potential is then

$$\begin{aligned}U &= \tfrac{1}{2}k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}k\,\big(d_{23} - \sqrt{2}\,a\big)^2 + \tfrac{1}{2}k\,\big(d_{13} - a\big)^2\\ &= \tfrac{1}{2}k\big(x_2 - x_1\big)^2 + \tfrac{1}{4}k\big(x_3 - x_2\big)^2 + \tfrac{1}{4}k\big(y_3 - y_2\big)^2\\ &\qquad - \tfrac{1}{2}k\big(x_3 - x_2\big)\big(y_3 - y_2\big) + \tfrac{1}{2}k\big(y_3 - y_1\big)^2\end{aligned}\tag{6.132}$$

Defining the row vector

$$\boldsymbol{\eta}^{\mathsf{t}} \equiv \big(x_1\ ,\ y_1\ ,\ x_2\ ,\ y_2\ ,\ x_3\ ,\ y_3\big)\quad ,\tag{6.133}$$

we have that U is a quadratic form:

$$U = \tfrac{1}{2}\eta_\sigma \mathsf{V}_{\sigma\sigma'}\eta_{\sigma'} = \tfrac{1}{2}\boldsymbol{\eta}^{\mathsf{t}}\,\mathsf{V}\,\boldsymbol{\eta},\tag{6.134}$$

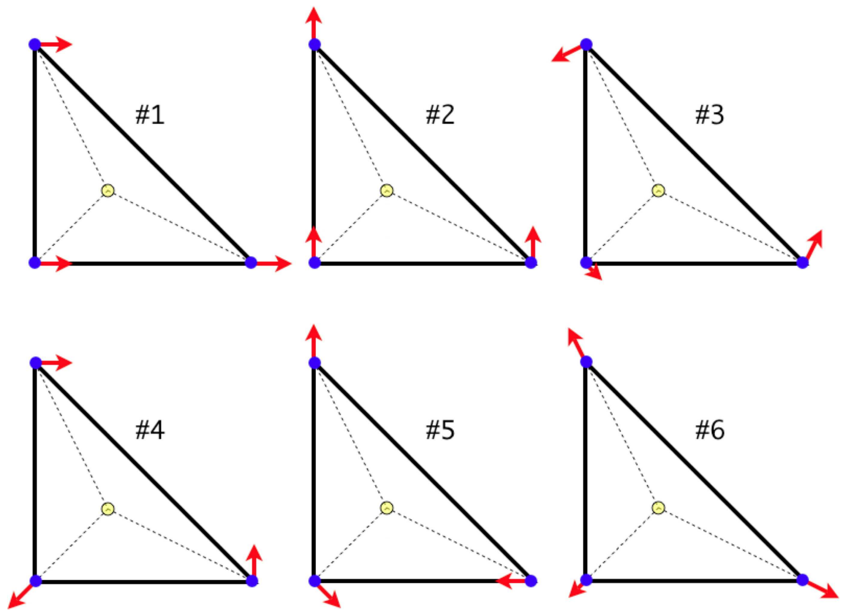

Figure 6.4: Normal modes of the $45°$ right triangle. The yellow circle is the location of the CM of the triangle. The labels for the vertices are 1 (lower left), 2 (lower right), and 3 (upper left).

with

$$
\mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{\bar{q}} = k \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \tag{6.135}
$$

The kinetic energy is simply

$$
T = \tfrac{1}{2} m \big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \quad , \tag{6.136}
$$

hence

$$
\mathsf{T}_{\sigma\sigma'} = m\,\delta_{\sigma\sigma'} \quad . \tag{6.137}
$$

(b) The three zero modes correspond to x-translation, y-translation, and rotation. Their eigenvectors, respectively, are

$$\psi_1 = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \psi_2 = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \psi_3 = \frac{1}{2\sqrt{3m}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \\ -2 \\ -1 \end{pmatrix} \quad . \tag{6.138}$$

Thus $\omega_1 = \omega_2 = \omega_3 = 0$. To find the unnormalized rotation vector, we find the CM of the triangle, located at $\left(\frac{a}{3}\,,\,\frac{a}{3}\right)$, and sketch orthogonal displacements $\hat{\boldsymbol{z}} \times (\boldsymbol{R}_a - \boldsymbol{R}_{\rm CM})$ at the position of mass point a.

(c) The remaining modes may be determined by symmetry, and are given by

$$\psi_4 = \frac{1}{2\sqrt{m}} \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \psi_5 = \frac{1}{2\sqrt{m}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \psi_6 = \frac{1}{2\sqrt{3m}} \begin{pmatrix} -1 \\ -1 \\ 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \quad , \tag{6.139}$$

with

$$\omega_4 = \sqrt{\frac{k}{m}} \quad , \quad \omega_5 = \sqrt{\frac{2k}{m}} \quad , \quad \omega_6 = \sqrt{\frac{3k}{m}} \quad . \tag{6.140}$$

Since $\mathsf{T} = m \cdot 1$ is a multiple of the unit matrix, the orthogonormality relation $\psi^{(i)}_\sigma \, \mathsf{T}_{\sigma\sigma'} \, \psi^{(j)}_{\sigma'} = \delta^{ij}$ entails $\psi^{(i)} \cdot \psi^{(j)} = m^{-1} \, \delta_{ij}$, *i.e.*the eigenvectors are mutually orthogonal in the conventional dot product sense. One can check that the eigenvectors listed here satisfy this condition.

The simplest of the set $\{\psi_4, \psi_5, \psi_6\}$ to find is the uniform dilation ψ_6, sometimes called the *breathing mode*. This must keep the triangle in the same shape, which means that the deviations at each mass point are proportional to the distance to the CM. Next simplest to find is ψ_4, in which the long and short sides of the triangle oscillate out of phase. Finally, the mode ψ_5 must be orthogonal to all the remaining modes. No heavy lifting (*e.g.* `Mathematica`) is required!

6.9.2 Triple pendulum

Consider a triple pendulum consisting of three identical masses m and three identical rigid massless rods of length ℓ, as depicted in fig. 6.5.

(a) Find the T and V matrices.

(b) Find the equation for the eigenfrequencies.

(c) Numerically solve the eigenvalue equation for ratios ω_j^2/ω_0^2, where $\omega_0 = \sqrt{g/\ell}$. Find the three normal modes.

Solution

The Cartesian coordinates for the three masses are

$$
\begin{aligned}
x_1 &= \ell\,\sin\theta_1 & y_1 &= -\ell\,\cos\theta_1 & (6.141)\\
x_2 &= \ell\,\sin\theta_1 + \ell\,\sin\theta_2 & y_2 &= -\ell\,\cos\theta_1 - \ell\,\cos\theta_2 & (6.142)\\
x_3 &= \ell\,\sin\theta_1 + \ell\,\sin\theta_2 + \ell\,\sin\theta_3 & y_3 &= -\ell\,\cos\theta_1 - \ell\,\cos\theta_2 - \ell\,\cos\theta_3 \quad . & (6.143)
\end{aligned}
$$

By inspection, we can write down the kinetic energy:

$$
\begin{aligned}
T &= \tfrac{1}{2}m\big(\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2 + \dot x_3^2 + \dot y_3^2\big)\\
&= \tfrac{1}{2}m\,\ell^2\left\{3\dot\theta_1^2 + 2\dot\theta_2^2 + \dot\theta_3^2 + 4\cos(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 + 2\cos(\theta_1-\theta_3)\,\dot\theta_1\,\dot\theta_3 + 2\cos(\theta_2-\theta_3)\,\dot\theta_2\,\dot\theta_3\right\}
\end{aligned} \tag{6.144}
$$

The potential energy is

$$
U = -mg\ell\left\{3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3\right\} \quad , \tag{6.145}
$$

and the Lagrangian is $L = T - U$:

$$
\begin{aligned}
L = \tfrac{1}{2}m\,\ell^2\Big\{3\,\dot\theta_1^2 &+ 2\,\dot\theta_2^2 + \dot\theta_3^2 + 4\cos(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 + 2\cos(\theta_1-\theta_3)\,\dot\theta_1\,\dot\theta_3\\
&+ 2\cos(\theta_2-\theta_3)\,\dot\theta_2\,\dot\theta_3\Big\} + mg\ell\left\{3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3\right\} \quad .
\end{aligned} \tag{6.146}
$$

The canonical momenta are given by

$$
\begin{aligned}
\pi_1 &= \frac{\partial L}{\partial\dot\theta_1} = m\,\ell^2\left\{3\dot\theta_1 + 2\cos(\theta_1-\theta_2)\,\dot\theta_2 + \cos(\theta_1-\theta_3)\,\dot\theta_3\right\}\\
\pi_2 &= \frac{\partial L}{\partial\dot\theta_2} = m\,\ell^2\left\{2\dot\theta_2 + 2\cos(\theta_1-\theta_2)\,\dot\theta_1 + \cos(\theta_2-\theta_3)\,\dot\theta_3\right\}\\
\pi_3 &= \frac{\partial L}{\partial\dot\theta_2} = m\,\ell^2\left\{\dot\theta_3 + \cos(\theta_1-\theta_3)\,\dot\theta_1 + \cos(\theta_2-\theta_3)\,\dot\theta_2\right\} \quad .
\end{aligned} \tag{6.147}
$$

The only conserved quantity is the total energy, $E = T + U$.

(a) As for the T and V matrices, we have

$$
\mathsf{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = m\ell^2\begin{pmatrix}3 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 1\end{pmatrix} \tag{6.148}
$$

and

$$
\mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = mg\ell\begin{pmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix} \quad . \tag{6.149}
$$

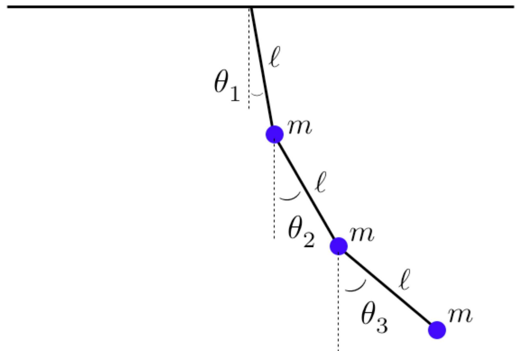

Figure 6.5: The triple pendulum.

(b) The eigenfrequencies are roots of the equation $\det\,(\omega^2\,\mathsf{T} - \mathsf{V}) = 0$. Defining $\omega_0 \equiv \sqrt{g/\ell}$, we have

$$\omega^2\,\mathsf{T} - \mathsf{V} = m\ell^2 \begin{pmatrix} 3(\omega^2 - \omega_0^2) & 2\,\omega^2 & \omega^2 \\ 2\,\omega^2 & 2(\omega^2 - \omega_0^2) & \omega^2 \\ \omega^2 & \omega^2 & (\omega^2 - \omega_0^2) \end{pmatrix} \tag{6.150}$$

and hence

$$\begin{aligned} \widetilde{P}(\omega) \equiv \det\,(\omega^2\mathsf{T} - \mathsf{V})/(m\ell^2)^3 &= 3(\omega^2 - \omega_0^2)\cdot\Big[2(\omega^2-\omega_0^2)^2 - \omega^4\Big] \\ &\qquad - 2\,\omega^2\cdot\Big[2\,\omega^2(\omega^2-\omega_0^2) - \omega^4\Big] + \omega^2\cdot\Big[2\,\omega^4 - 2\,\omega^2(\omega^2-\omega_0^2)\Big] \\ &= 6\,(\omega^2-\omega_0^2)^3 - 9\,\omega^4\,(\omega^2-\omega_0^2) + 4\omega^6 \\ &= \omega^6 - 9\,\omega_0^2\,\omega^4 + 18\,\omega_0^4\,\omega^2 - 6\,\omega_0^6 \quad . \end{aligned} \tag{6.151}$$

(c) The equation for the eigenfrequencies is

$$\lambda^3 - 9\,\lambda^2 + 18\,\lambda - 6 = 0 \quad , \tag{6.152}$$

where $\omega^2 = \lambda\,\omega_0^2$. This is a cubic equation in λ. Numerically solving for the roots, one finds

$$\omega_1^2 = 0.415774\,\omega_0^2 \quad , \quad \omega_2^2 = 2.29428\,\omega_0^2 \quad , \quad \omega_3^2 = 6.28995\,\omega_0^2 \quad . \tag{6.153}$$

I find the (unnormalized) eigenvectors to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} 1 \\ 1.2921 \\ 1.6312 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \begin{pmatrix} 1 \\ 0.35286 \\ -2.3981 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \begin{pmatrix} 1 \\ -1.6450 \\ 0.76690 \end{pmatrix} \quad . \tag{6.154}$$

6.9.3 Equilateral linear triatomic molecule

Consider the vibrations of an equilateral triangle of mass points, depicted in figure 6.6 . The system is confined to the (x, y) plane, and in equilibrium all the strings are unstretched and of length a.

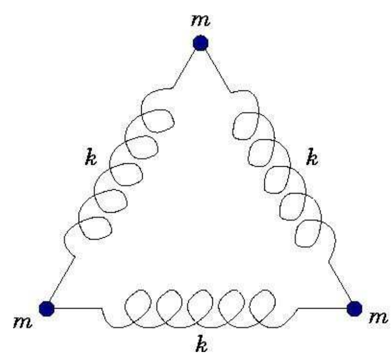

Figure 6.6: An equilateral triangle of identical mass points and springs. We label the sites as 1 (lower left), 2 (lower right), and 3 (upper).

(a) Choose as generalized coordinates the Cartesian displacements (x_i, y_i) with respect to equilibrium. Write down the exact potential energy.

(b) Find the T and V matrices.

(c) There are three normal modes of oscillation for which the corresponding eigenfrequencies all vanish: $\omega_a = 0$. Write down these modes explicitly, and provide a physical interpretation for why $\omega_a = 0$. Since this triplet is degenerate, there is no unique answer – any linear combination will also serve as a valid 'zero mode'. However, if you think physically, a natural set should emerge.

(d) The three remaining modes all have finite oscillation frequencies. They correspond to distortions of the triangular shape. One such mode is the "breathing mode" in which the triangle uniformly expands and contracts. Write down the eigenvector associated with this normal mode and compute its associated oscillation frequency.

(e) The fifth and sixth modes are degenerate. They must be orthogonal (with respect to the inner product defined by T) to all the other modes. See if you can figure out what these modes are, and compute their oscillation frequencies. As in (a), any linear combination of these modes will also be an eigenmode.

(f) Write down the modal matrix $\mathsf{A}_{\sigma i}$, and check that it is correct by using `Mathematica`.

Solution

Choosing as generalized coordinates the Cartesian displacements relative to equilibrium, we have the following:

$$\begin{aligned}
\#1 &: \big(x_1, y_1\big) \\
\#2 &: \big(a + x_2, y_2\big) \\
\#3 &: \big(\tfrac{1}{2}a + x_3, \tfrac{\sqrt{3}}{2}a + y_3\big) \quad .
\end{aligned}$$

Let d_{ij} be the separation of particles i and j. The potential energy of the spring connecting them is then $\frac{1}{2}k(d_{ij}-a)^2$.

$$\begin{aligned} d_{12}^2 &= \big(a + x_2 - x_1\big)^2 + \big(y_2 - y_1\big)^2 \\ d_{23}^2 &= \big(-\tfrac{1}{2}a + x_3 - x_2\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_2\big)^2 \\ d_{13}^2 &= \big(\tfrac{1}{2}a + x_3 - x_1\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_1\big)^2 \quad . \end{aligned} \tag{6.155}$$

The full potential energy is

$$U = \tfrac{1}{2}\,k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{23} - a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{13} - a\big)^2 \quad . \tag{6.156}$$

This is a cumbersome expression, involving square roots.

To find T and V, we need to write T and V as quadratic forms, neglecting higher order terms. Therefore, we must expand $d_{ij} - a$ to linear order in the generalized coordinates. This results in the following:

$$\begin{aligned} d_{12} &= a + \big(x_2 - x_1\big) + \ldots \\ d_{23} &= a - \tfrac{1}{2}\,\big(x_3 - x_2\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_2\big) + \ldots \\ d_{13} &= a + \tfrac{1}{2}\,\big(x_3 - x_1\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_1\big) + \ldots \quad . \end{aligned} \tag{6.157}$$

Thus,

$$\begin{aligned} U = \tfrac{1}{2}k\big(x_2 - x_1\big)^2 &+ \tfrac{1}{8}k\big(x_2 - x_3 - \sqrt{3}\,y_2 + \sqrt{3}\,y_3\big)^2 \\ &+ \tfrac{1}{8}k\big(x_3 - x_1 + \sqrt{3}\,y_3 - \sqrt{3}\,y_1\big)^2 + \text{higher order terms} \quad . \end{aligned} \tag{6.158}$$

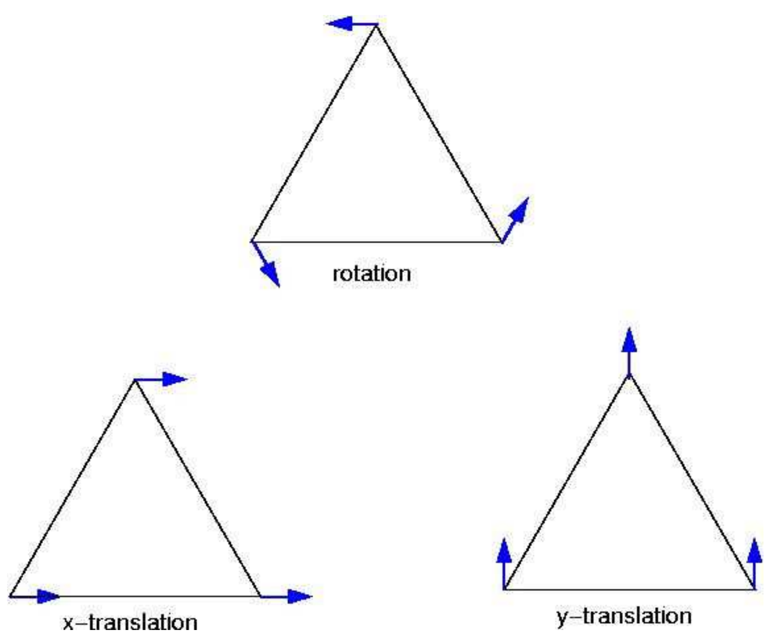

Figure 6.7: Zero modes of the mass-spring triangle.

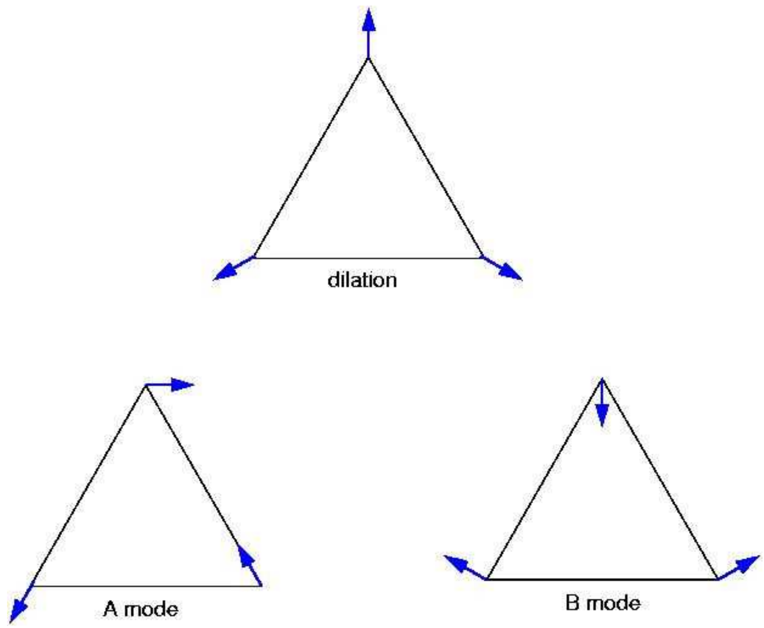

Figure 6.8: Finite oscillation frequency modes of the mass-spring triangle.

Defining

$$\big(q_1, q_2, q_3, q_4, q_5, q_6\big) = \big(x_1, y_1, x_2, y_2, x_3, y_3\big) \quad , \tag{6.159}$$

we may now read off

$$\mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma \, \partial q_{\sigma'}}\bigg|_{\bar q} = k \begin{pmatrix} 5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 \\ -1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ 0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2 \end{pmatrix} \tag{6.160}$$

The T matrix is trivial. From

$$T = \tfrac{1}{2} m \big(\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2 + \dot x_3^2 + \dot y_3^2 \big) \quad . \tag{6.161}$$

we obtain

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial \dot q_\sigma \, \partial \dot q_{\sigma'}} = m \, \delta_{\sigma\sigma'} \quad , \tag{6.162}$$

and $\mathsf{T} = m \cdot \mathbb{I}$ is a multiple of the unit matrix.

Figure 6.9: *John Henry*, statue by Charles O. Cooper (1972). "Now the man that invented the steam drill, he thought he was mighty fine. But John Henry drove fifteen feet, and the steam drill only made nine." - from *The Ballad of John Henry*.

The zero modes are depicted graphically in figure 6.7. Explicitly, we have

$$\boldsymbol{\xi}^x = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}^y = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}^{\rm rot} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ -1 \\ 0 \end{pmatrix} \quad . \tag{6.163}$$

That these are indeed zero modes may be verified by direct multiplication:

$$\mathsf{V}\,\boldsymbol{\xi}^{x.y} = \mathsf{V}\,\boldsymbol{\xi}^{\rm rot} = 0 \quad . \tag{6.164}$$

The three modes with finite oscillation frequency are depicted graphically in figure 6.8. Explicitly, we have

$$\boldsymbol{\xi}^{\rm A} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}^{\rm B} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ 1/2 \\ 0 \\ -1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}^{\rm dil} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix} \quad . \tag{6.165}$$

The oscillation frequencies of these modes are easily checked by multiplying the eigenvectors by the matrix V. Since $\mathsf{T} = m \cdot \mathbb{I}$ is diagonal, we have $\mathsf{V}\,\boldsymbol{\xi}^{(j)} = m\omega_j^2\,\boldsymbol{\xi}^{(j)}$. One finds

$$\omega_{\rm A} = \omega_{\rm B} = \sqrt{\frac{3k}{2m}} \quad , \quad \omega_{\rm dil} = \sqrt{\frac{3k}{m}} \quad . \tag{6.166}$$

`Mathematica`? I don't need no stinking `Mathematica`.

6.10 Aside: Christoffel Symbols

The coupled equations in eqn. 6.5 may be written in the form

$$\ddot{q}_\sigma + \Gamma^\sigma_{\mu\nu}\,\dot{q}_\mu\,\dot{q}_\nu = W_\sigma \quad , \tag{6.167}$$

with

$$\Gamma^\sigma_{\mu\nu} = \tfrac{1}{2}\,T^{-1}_{\sigma\alpha}\bigg(\frac{\partial T_{\alpha\mu}}{\partial q_\nu} + \frac{\partial T_{\alpha\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\alpha}\bigg) \tag{6.168}$$

and

$$W_\sigma = -T^{-1}_{\sigma\alpha}\,\frac{\partial U}{\partial q_\alpha} \quad . \tag{6.169}$$

The components of the rank-three tensor $\Gamma^\sigma_{\alpha\beta}$ are known as *Christoffel symbols*, in the case where $T_{\mu\nu}(q)$ defines a *metric* on the space of generalized coordinates.