1. $\psi(x) = Axe^{-bx}$ gives $d\psi/dx = Ae^{-bx} - bAxe^{-bx}$ and $d^2\psi/dx^2 = -2bAe^{-bx} + b^2 Axe^{-bx}$. Then substituting into Equation 7.2 we have

$$-\frac{\hbar^2}{2m}(-2Abe^{-bx} + b^2 Axe^{-bx}) - \frac{e^2}{4\pi\varepsilon_0 x} Axe^{-bx} = EAxe^{-bx}$$

Canceling common factors gives

$$\frac{\hbar^2 b}{m} - \frac{\hbar^2 b^2}{2m} x - \frac{e^2}{4\pi\varepsilon_0} = Ex \quad \text{or} \quad \left(\frac{\hbar^2 b}{m} - \frac{e^2}{4\pi\varepsilon_0}\right) + x\left(-\frac{\hbar^2 b^2}{2m} - E\right) = 0$$

For this expression to equal zero for all *x*, both terms in parentheses must be zero. Thus

$$\frac{\hbar^2 b}{m} = \frac{e^2}{4\pi\varepsilon_0} \quad \text{or} \quad b = \frac{me^2}{4\pi\varepsilon_0\hbar^2} = \frac{1}{a_0} \qquad \text{and} \qquad E = -\frac{\hbar^2 b^2}{2m} = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}$$

2. The probability density is $P(x) = |\psi(x)|^2 = A^2 x^2 e^{-2bx}$. To find the maximum, we set the first derivative equal to zero:

$$\frac{dP}{dx} = 2A^2 xe^{-2bx} - 2bA^2 x^2 e^{-2bx} = 0$$

This has solutions at $x = 0$, $x = \infty$, and $x = 1/b = a_0$. The first two give minima and the third gives the maximum.

3. The probability to find the electron in a small interval is $P(x)dx = A^2 x^2 e^{-2bx} dx$. Substituting the values of *A* and *b*, and evaluating the resulting expression for $x = a_0$ and $dx = 0.02a_0$ (appropriate to the interval from $x = 0.99a_0$ to $x = 1.01a_0$), we obtain

$$P(x)dx = \frac{4}{a_0^3} x^2 e^{-2x/a_0} dx = \frac{4}{a_0^3} a_0^2 e^{-2} (0.02a_0) = 0.0108$$

4. (a) $|\mathbf{L}| = \sqrt{l(l+1)}\hbar = \sqrt{(3)(4)}\hbar = \sqrt{12}\hbar$

(b) There are $2l + 1 = 7$ possible *z* components: $L_z = m_l\hbar = +3\hbar, +2\hbar, +\hbar, 0, -\hbar, -2\hbar, -3\hbar$.

(c) $\cos\theta = m_l/\sqrt{l(l+1)} = m_l/\sqrt{12}$

$$m_l = +3 \qquad \theta = \cos^{-1} 3/\sqrt{12} = 30°$$

$$m_l = +2 \qquad \theta = \cos^{-1} 2/\sqrt{12} = 55°$$

$$m_l = +1 \qquad \theta = \cos^{-1} 1/\sqrt{12} = 73°$$

$$m_l = 0 \qquad \theta = \cos^{-1} 0 = 90°$$

$$m_l = -1 \qquad \theta = \cos^{-1}(-1/\sqrt{12}) = 107°$$

$$m_l = -2 \qquad \theta = \cos^{-1}(-2/\sqrt{12}) = 125°$$

$$m_l = -3 \qquad \theta = \cos^{-1}(-3/\sqrt{12}) = 150°$$

7. (a) $l_{max} = n - 1 = 5$ so $l = 0, 1, 2, 3, 4, 5$ for $n = 6$.

(b) $m_l = +6, +5, +4, +3, +2, +1, 0, -1, -2, -3, -4, -5, -6$

(c) $n \geq l + 1 = 5$ for $l = 4$, so the smallest possible n is 5.

(d) For $m_l = 4$, $l \geq 4$ so the smallest possible l is 4.

10. With $\psi_{1,0,0}(r,\theta,\phi)=\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$, $\frac{\partial\psi}{\partial r}=\frac{1}{\sqrt{\pi a_0^3}}\left(-\frac{1}{a_0}\right)e^{-r/a_0}$ and $\frac{\partial^2\psi}{\partial r^2}=\frac{1}{\sqrt{\pi a_0^3}}\left(\frac{1}{a_0^2}\right)e^{-r/a_0}$.

Substituting into Equation 7.10, we have

$$\frac{1}{\sqrt{\pi a_0^3}}\left[-\frac{\hbar^2}{2m}\left(\frac{1}{a_0^2}e^{-r/a_0}-\frac{2}{a_0 r}e^{-r/a_0}\right)-\frac{e^2}{4\pi\varepsilon_0 r}e^{-r/a_0}\right]$$

$$=\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\frac{e^2}{4\pi\varepsilon_0}\left(-\frac{1}{2a_0}+\frac{1}{r}-\frac{1}{r}\right)=-\frac{1}{2a_0}\frac{e^2}{4\pi\varepsilon_0}\psi_{1,0,0}(r,\theta,\phi)=E\psi_{1,0,0}(r,\theta,\phi)$$

with $E=-\frac{1}{2a_0}\frac{e^2}{4\pi\varepsilon_0}=-\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{me^2}{4\pi\varepsilon_0\hbar^2}=-\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}$ which is E_1 from Equation 7.13.