PHYSICS 200B : CLASSICAL MECHANICS
FINAL EXAM SOLUTIONS

[1] Consider the time-dependent 'kicked' Hamiltonian $H(t) = T(p) + V(q)\,K(t)$, where $K(t) = \tau \sum_n \delta(t - n\tau)$ is a Dirac comb. Let $q_n = q(n\tau^-)$ and $p_n = p(n\tau^-)$, *i.e.* just before each kick.

(a) Find the matrix

$$M_n = \frac{\partial(q_{n+1}, p_{n+1})}{\partial(q_n, p_n)} \quad ,$$

and show that it is symplectic.

Hamilton's equations are $\dot{q} = -T'(p)$ and $\dot{p} = V'(q)\,K(t)$. Integrating from $t = n\tau^-$ to $t = (n+1)\tau^-$, we obtain

$$q_{n+1} = q_n - \tau\, T'(p_{n+1}) \quad , \quad p_{n+1} = p_n + \tau\, V'(q_n) \quad .$$

Thus,

$$\begin{aligned} dp_{n+1} &= dp_n + \tau\, V''(q_n)\, dq_n \\ dq_{n+1} &= dq_n - \tau\, T''(p_{n+1})\, dp_{n+1} \\ &= \Big[1 - \tau^2\, T''(p_{n+1})\, V''(q_n)\Big] dq_n - \tau\, T''(p_{n+1})\, dp_n \quad . \end{aligned}$$

Thus,

$$M_n = \frac{\partial(q_{n+1}, p_{n+1})}{\partial(q_n, p_n)} = \begin{pmatrix} 1 - \tau^2\, T''(p_{n+1})\, V''(q_n) & -\tau\, T''(p_{n+1}) \\ \tau\, V''(q_n) & 1 \end{pmatrix}$$

Now consider a general 2×2 matrix, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$M^{\mathsf{t}}\, J\, M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad .$$

Thus, any 2×2 matrix M with unit determinant $\mathsf{det}\, M = ad - bc$ is symplectic. Note that $\mathsf{det}\, M_n = 1$, hence M_n is symplectic.

(b) Find the condition that a fixed point (q^*, p^*) is unstable.

The matrix M_n is of the form $M_n = \begin{pmatrix} 1 - ab & -a \\ b & 1 \end{pmatrix}$, where $a = \tau\, T''(p_{n+1})$ and $b = \tau\, V''(q_n)$. The characteristic polynomial of any 2×2 matrix M is $P(\lambda) = \mathsf{det}(\lambda - M) = \lambda^2 - T\lambda + D$ where $T = \mathsf{Tr}\, M$ and $D = \mathsf{det}\, M$. Since $D = 1$ we have $P(\lambda) = \lambda^2 - T\lambda + 1$, with roots $\lambda_\pm = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2 - 4}$ with $T = 2 - ab$. Note $\lambda_+ \lambda_- = D = 1$. For $T^2 < 4$ the eigenvalues are of the form $\lambda_\pm = e^{\pm i\theta}$ with $\theta = \cos^{-1}(T/2)$. When $T^2 > 4$ we have $\lambda_\pm = \mathsf{sgn}(T)\, e^{\pm\beta}$, with $\beta = \cosh^{-1}(|T|/2) = \log\big(\frac{1}{2}|T| + \frac{1}{2}\sqrt{T^2 - 4}\big)$. Thus, stability requires $T \in [-2, 2]$, *i.e.*

$$0 < \tau^2\, T''(p^*)\, V''(q^*) < 4 \quad .$$

The condition for instability is that this condition is violated, *i.e.* either $\tau^2\, T''(p^*)\, V''(q^*) < 0$ or $\tau^2\, T''(p^*)\, V''(q^*) > 4$.

(c) Define the function

$$g(x) = x - \mathsf{nint}(x) \quad ,$$

where $\mathsf{nint}(x)$ is the nearest integer to x. Thus $g(\pm 0.4) = \pm 0.4$ since $\mathsf{nint}(\pm 0.4) = 0$, but $g(0.6) = -0.4$, $g(-3.7) = 0.3$, *etc.* Now consider the case

$$T(p) = \frac{P^2}{2m} \cdot \big[g(p/P)\big]^2 \qquad , \qquad V(q) = \tfrac{1}{2} k Q^2 \cdot \big[g(q/Q)\big]^2 \quad .$$

This effectively renders the phase space a torus of area PQ. Find the conditions for all fixed points of the map $(q_n, p_n) \to (q_{n+1}, p_{n+1})$. Which fixed points are unstable?

We have, for $q \in [-\frac{1}{2}Q, \frac{1}{2}Q]$ and $p \in [-\frac{1}{2}P, \frac{1}{2}P]$,

$$T'(p) = \frac{p}{m} \qquad , \qquad V'(q) = kq \quad .$$

Note that $T''(p) = m^{-1}$ and $V''(q) = k$ independent of (q, p). Thus, $T = 2 - \omega^2\tau^2$, where $\omega^2 = k/m$, and the instability condition is $|\omega\tau| > 2$ for all fixed points. To find the fixed points, set $q_n = Q x_n$ and $p_n = P y_n$. The map is

$$\begin{aligned} x_{n+1} &= \big(1 - \omega^2\tau^2\big)\, x_n - r\,\omega\tau\, y_n \ \mathsf{mod}\ 1 \\ y_{n+1} &= y_n + r^{-1}\omega\tau\, x_n \ \mathsf{mod}\ 1 \quad , \end{aligned}$$

where $r = P/Q\sqrt{mk}$. Here,"mod 1" folds each of x_{n+1} and y_{n+1} into the interval $[-\frac{1}{2}, \frac{1}{2}]$. So consider a fixed point (x^*, y^*). We must have

$$\begin{aligned} r^{-1}\omega\tau x^* &= j \\ \omega^2\tau^2 x^* + r\omega\tau y^* &= l \quad , \end{aligned}$$

where $|x^*| < \frac{1}{2}$ and $|y^*| < \frac{1}{2}$ and both j and l are integers. One obvious solution is $j = l = 0$, yielding $x^* = y^* = 0$. But there may be others, depending on the values of $\omega\tau$ and r. The general expression for fixed points is then

$$x^* = j \cdot \frac{r}{\omega\tau} \quad , \quad y^* = l \cdot \frac{1}{r\omega\tau} - j \cdot \frac{r}{\omega\tau} \quad .$$

[2] Consider the 1D map $x_{n+1} = f(x_n)$, where

$$f(x) = rx(1-x)(1-2x)^2 \quad .$$

(a) Numerically explore the stability of the fixed 1-cycle by plotting cobweb diagrams for various values of r. Note that $f(x) = f(1-x)$, $f'(0) = r$, but $f(\frac{1}{2}) = 0$. Thus, as r changes, new solutions to the fixed point equation $f(x) = x$ may appear discontinuously. Can you numerically identify the ranges of stability?

See fig 2.

(b) Another way to investigate is the following. Write a computer program which makes a plot like in fig. 2.10 of the lecture notes. Here is how I made that figure:

i. The outer loop is over the r values. For this problem, choose $r \in [1, 16]$. Loop over at least 500 values.

ii. For each r value, iterate the map $x' = f(x)$ one thousand times, but do not plot the results. Start with a random seed x_0. (You can even try using the same seed for each r value.)

iii. After iterating so many times, your program should have settled in on a stable cycle or else it is in a regime of chaos. Plot the next 400 iterates of the map.

iv. Advance r to its next value $r + \Delta r$ and go back to step (ii). Terminate after $r = 16$.

See fig 1.

(c) Analytically obtain the region of stability in the control parameter r and the corresponding set of fixed points $x^*(r)$. *Hint: Simultaneously set $f(x) = x$ and $f'(x) = \pm 1$.*

Taking the derivative, we find

$$f'(x) = r(1 - 2x)(1 - 8x + 8x^2) \quad ,$$

which is cubic. We also have

$$\frac{f(x)}{x} = r(1 - x)(1 - 2x)^2 \quad ,$$

which is also cubic. Setting $f(x) = x$ and $f'(x) = \pm 1$, we have

$$r(1 - x)(1 - 2x)^2 = \pm r(1 - 2x)(1 - 8x + 8x^2) = 1 \quad .$$

Note that the first two expressions share the common factor $(1 - 2x)$. Dividing by this factor yields a quadratic equation! Taking the upper $(+)$ sign, we obtain

$$(1 - x)(1 - 2x) = 1 - 8x + 8x^2 \quad \Rightarrow \quad 6x^2 - 5x - 0 \quad ,$$

with the solutions $x = 0$ and $x = \frac{5}{6}$. Plugging these values into either of the equations $f(x) = x$ or $f'(x) = +1$ yields $r(0) = 1$ and $r(\frac{5}{6}) = \frac{27}{2}$. Now consider the lower sign $(-)$ case where $f'(x) = -1$. We then obtain the quadratic equation $1 - x^2 - 11x + 2 = 0$, with solutions

$$x_{\pm} = \tfrac{1}{20}\big(11 \pm \sqrt{41}\big) \quad .$$

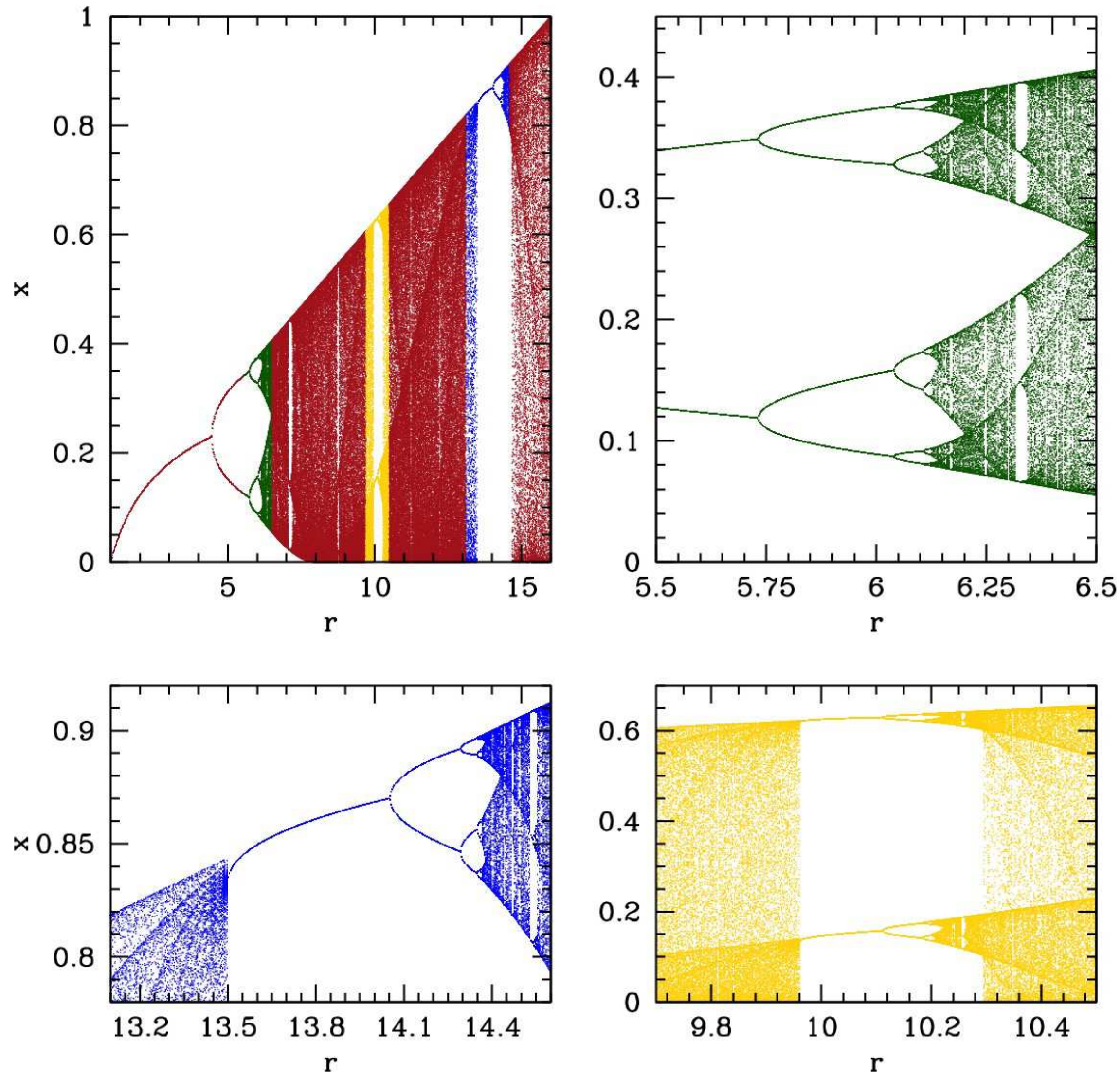

Figure 1: Iterations of the map $x' = rx(1-x)(1-2x)^2$.

Numerically, $x_- = 0.2298$ and $x_+ = 0.8702$. Plugging these values into $f(x) = x$ or $f'(x) = 1$, we find $r(0.2298) = 4.448$ and $r(0.8702) = 14.05$. Thus, there are two regions where there is a stable fixed point (1-cycle): (i) $r \in [1, 4.448]$ and (ii) $r \in [13.5, 14.05]$.

(d) Show that for $r = 16$, if we define $x \equiv \sin^2\theta$, with $\theta \in [0, \pi]$, there is a simple relationship between θ_{n+1} and θ_n. Writing the binary expansion of $\theta_{n=0}$ as

$$\theta_0 = \pi \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad ,$$

and given that $\sin^2\theta$ is periodic under $\theta \to \theta + \pi$, find the corresponding binary expansion of θ_n.

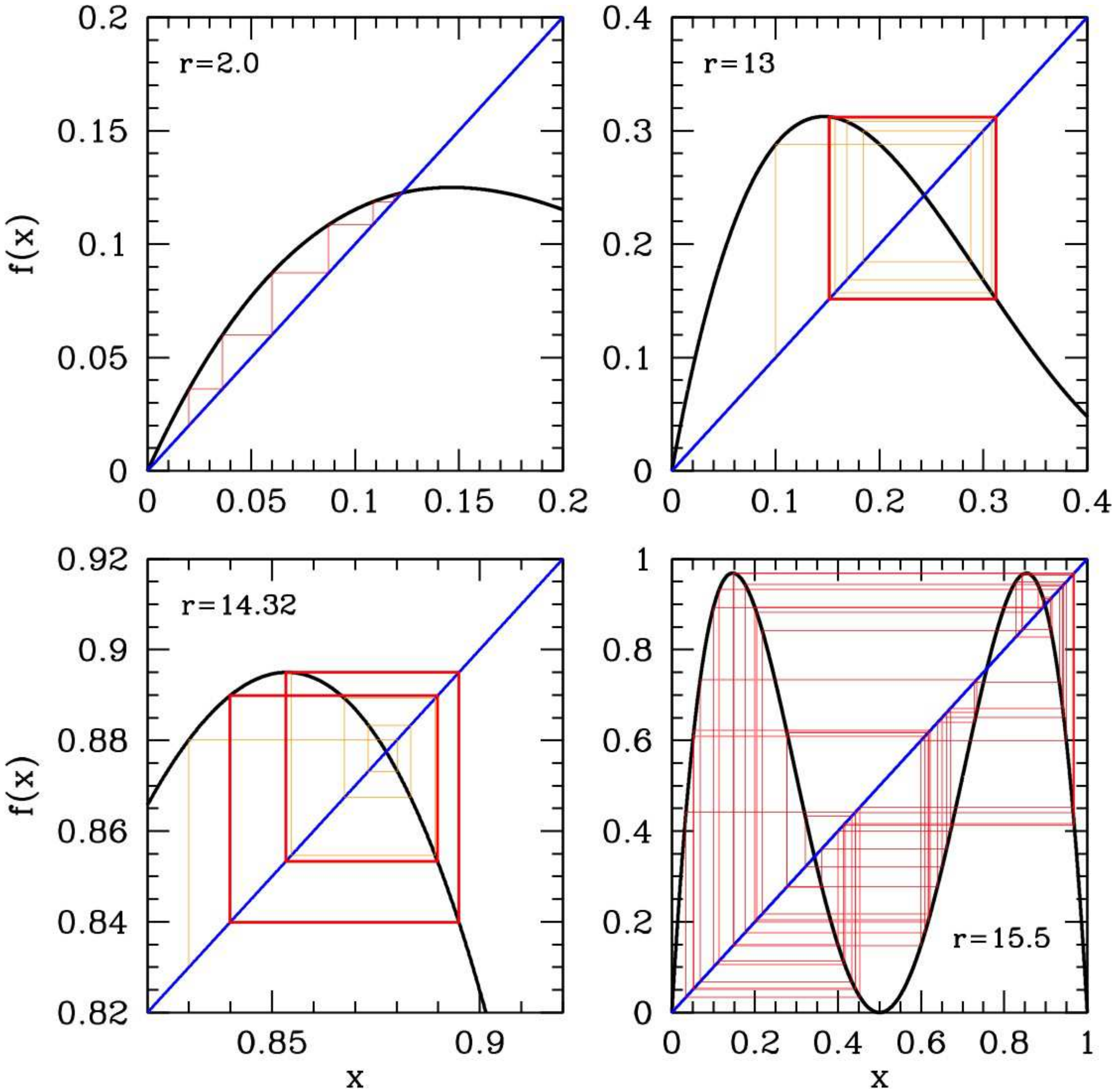

Figure 2: Cobweb diagrams for the map $x' = rx(1-x)(1-2x)^2$.

With $x \equiv \sin^2\theta$, we have

$$f(x) = rx(1-x)(1-2x)^2 = r\sin^2\!\theta\cos^2\!\theta\cos^2\!(2\theta) = \tfrac{1}{16}\,r\sin^2\!(4\theta) \quad .$$

Hence when $r = 16$ we have the map $\theta_{n+1} = 4\theta_n$ and writing $\theta_0 = \pi\sum_{k=1}^{\infty} 2^{-k}\,b_k$ we obtain

$$\theta_n = \pi\sum_{k=1}^{\infty}\frac{b_{k+2n}}{2^k} \quad .$$

That is, each iteration of the map at $r = 16$ shifts the binary expansion of θ two digits to the left.

[3] *The Burgers vortex* – Seek an exact, steady state solution to the Navier-Stokes equations (with $\zeta = 0$) of the form

$$\boldsymbol{v}(r,\phi,z) = -\tfrac{1}{2}\alpha r\hat{\mathbf{e}}_r + v_\phi(r)\,\hat{\mathbf{e}}_\phi + \alpha z\,\hat{\mathbf{e}}_z \quad .$$

(a) Verify that $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ and that $\boldsymbol{\omega} = \omega(r)\,\hat{\mathbf{e}}_z$. Show that the equations of motion imply a first order ODE for the vorticity $\omega(r)$. Obtain that equation.

First, some vector calculus relations in 3D cylindrical coordinates (r,ϕ,z). The gradient is

$$\boldsymbol{\nabla} U = \frac{\partial U}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\,\frac{\partial U}{\partial \phi}\,\hat{\mathbf{e}}_\phi + \frac{\partial U}{\partial z}\,\hat{\mathbf{e}}_z \quad .$$

The divergence is

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r}\,\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\,\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad .$$

The curl is

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \left(\frac{1}{r}\,\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \left(\frac{1}{r}\,\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{r}\,\frac{\partial A_r}{\partial \phi}\right)\hat{\mathbf{e}}_z \quad .$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \quad .$$

Thus, the divergence of the velocity field given above is

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}\big(-\tfrac{1}{2}\alpha r^2\big) + \frac{1}{r}\frac{\partial}{\partial \phi}\,v_\phi(r) + \frac{\partial}{\partial z}\,(\alpha z) = 0$$

The vorticity is

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v} = \frac{1}{r}\frac{\partial\big(r v_\phi(r)\big)}{\partial r}\,\hat{\mathbf{e}}_z = \left(\frac{dv_\phi(r)}{dr} + \frac{v_\phi(r)}{r}\right)\hat{\mathbf{e}}_z = \omega(r)\,\hat{\mathbf{e}}_z \quad .$$

Now from the NS equation we have

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) + \nu\,\nabla^2\omega(r)\,\hat{\mathbf{e}}_z = 0 \quad ,$$

since $\boldsymbol{\omega} = \omega(r)\,\hat{\mathbf{e}}_z$ is time-independent in the steady-state limit. Since $\boldsymbol{\omega} = \omega\,\hat{\mathbf{e}}_z$, we have

$$\boldsymbol{v}\times\boldsymbol{\omega} = \mathsf{det}\begin{pmatrix}\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ v_r & v_\phi & v_z \\ 0 & 0 & \omega\end{pmatrix} = \omega\,v_\phi(r)\,\hat{\mathbf{e}}_r + \tfrac{1}{2}\,\omega\,r\,\hat{\mathbf{e}}_\phi \quad ,$$

and

$$\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \frac{1}{r}\frac{\partial}{\partial r}\Big(\tfrac{1}{2}\alpha r^2\omega(r)\Big)\,\hat{\mathbf{e}}_z \quad .$$

Now

$$\nabla^2\omega = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\omega(r)\Big)\,\hat{\mathbf{e}}_z \quad .$$

Integrating the equation $\boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \nu\, \nabla^2 \omega(r)\, \hat{\mathbf{e}}_z = 0$ once, we have

$$\tfrac{1}{2}\alpha r \omega + \nu\, \frac{d\omega}{dr} = 0 \quad \Rightarrow \quad \frac{d\omega}{\omega} = -\frac{\alpha r}{2\nu} \quad .$$

(b) Find $\omega(r)$ and $v_\phi(r)$.

Integrating the first order ODE for $\omega(r)$, we have

$$\log \omega = \log C - \frac{\alpha r^2}{4\nu} \quad \Rightarrow \quad \omega(r) = C\, e^{-\alpha r^2/4\nu} \quad .$$

where C is a constant. From

$$\omega(r) = \frac{1}{r} \frac{d(r v_\phi)}{dr} = C\, e^{-\alpha r^2/4\nu} \quad .$$

Integrating, we have

$$v_\phi(r) = C' \Big(1 - e^{-\alpha r^2/4\nu} \Big) \quad ,$$

where $C' = 2\nu C/\alpha$, and where the second constant of integration is chosen so that $\varphi(r)$ is not divergent as $r \to 0$,