PHYSICS 200B : CLASSICAL MECHANICS
HOMEWORK SET #3

[**1**] Consider the matrix

$$M = \begin{pmatrix} 4 & 4 \\ -1 & 9 \end{pmatrix} \quad .$$

(a) Find the characteristic polynomial $P(\lambda) = \mathsf{det}(\lambda\mathbb{I} - M)$ and the eigenvalues.

(b) For each eigenvalue λ_α, find the associated right eigenvector R^α_i and left eigenvector L^α_i. Normalize your eigenvectors so that $\langle\, L^\alpha \,|\, R^\beta \,\rangle = \delta_{\alpha\beta}$.

(c) Show explicitly that $M_{ij} = \sum_\alpha \lambda_\alpha \, R^\alpha_i \, L^\alpha_j$.

[**2**] Consider a three-state system with the following transition rates:

$$W_{12} = 0 \quad , \quad W_{21} = \gamma \quad , \quad W_{23} = 0 \quad , \quad W_{32} = 3\gamma \quad , \quad W_{13} = \gamma \quad , \quad W_{31} = \gamma \quad .$$

(a) Find the matrix Γ such that $\dot{P}_i = -\Gamma_{ij} P_j$.

(b) Find the equilibrium distribution $P^{\rm eq}_i$.

(c) Does this system satisfy detailed balance? Why or why not?

[**3**] A *Markov chain* is a process which describes transitions of a discrete stochastic variable occurring at discrete times. Let $P_i(t)$ be the probability that the system is in state i at time t. The evolution equation is

$$P_i(t+1) = \sum_j Q_{ij} \, P_j(t) \quad .$$

The *transition matrix* Q_{ij} satisfies $\sum_i Q_{ij} = 1$ so that the total probability $\sum_i P_i(t)$ is conserved. The element Q_{ij} is the *conditional probability* that for the system to evolve to state i at time $t+1$ given that it was in state j at time t. Now consider a group of Physics graduate students consisting of three theorists and four experimentalists. Within each group, the students are to be regarded as indistinguishable. Together, the students rent two apartments, A and B. Initially the three theorists live in A and the four experimentalists live in B. Each month, a random occupant of A and a random occupant of B exchange domiciles. Compute the transition matrix Q_{ij} for this Markov chain, and compute the average fraction of the time that B contains two theorists and two experimentalists, averaged over the effectively infinite time it takes the students to get their degrees. *Hint:* Q is a 4×4 matrix.

[**4**] Consider a modified version of the Kac ring model where each spin exists in one of three states: A, B, or C. The flippers rotate the internal states cyclically: A$\to$B$\to$C$\to$A.

(a) What is the Poincaré recurrence time for this system? *Hint:* the answer depends on whether or not the total number of flippers is a multiple of 3.

(b) Simulate the system numerically. Choose a ring size on the order of $N = 10,000$ and investigate a few flipper densities: $x = 0.001$, $x = 0.01$, $x = 0.1$, $x = 0.99$. Remember that the flippers are located randomly at the start, but do not move as the spins evolve. Starting from a configuration where all the spins are in the A state, plot the probabilities $p_{\rm A}(t)$, $p_{\rm B}(t)$, and $p_{\rm C}(t)$ *versus* the discrete time coordinate t, with t ranging from 0 to the recurrence time. If you can, for each value of x, plot the three probabilities in different colors or line characteristics (*e.g.* solid, dotted, dashed) on the same graph.

(c) Let's call $a_t = p_{\rm A}(t)$, *etc.* Explain in words why the *Stosszahlansatz* results in the equations

$$
\begin{aligned}
a_{t+1} &= (1-x)\, a_t + x\, c_t \\
b_{t+1} &= (1-x)\, b_t + x\, a_t \\
c_{t+1} &= (1-x)\, c_t + x\, b_t \quad .
\end{aligned}
$$

This describes what is known as a *Markov process*, which is governed by coupled equations of the form $P_i(t+1) = \sum_j Q_{ij}\, P_j(t)$, where Q is the *transition matrix*. Find the 3×3 transition matrix for this Markov process.

(d) Show that the total probability is conserved by a Markov process if $\sum_i Q_{ij} = 1$ and verify this is the case for the equations in (c).

(e) One can then eliminate $c_t = 1 - a_t - b_t$ and write these as two coupled equations. Show that if we define

$$
\tilde{a}_t \equiv a_t - \tfrac{1}{3} \quad , \quad \tilde{b}_t \equiv b_t - \tfrac{1}{3} \quad , \quad \tilde{c}_t \equiv c_t - \tfrac{1}{3}
$$

that we can write

$$
\begin{pmatrix} \tilde{a}_{t+1} \\ \tilde{b}_{t+1} \end{pmatrix} = R \begin{pmatrix} \tilde{a}_t \\ \tilde{b}_t \end{pmatrix} \quad ,
$$

and find the 2×2 matrix R. Note that this is *not* a Markov process in A and B, since total probability for the A and B states is not itself conserved. Show that the eigenvalues of R form a complex conjugate pair. Find the amplitude and phase of these eigenvalues. Show that the amplitude never exceeds unity.

(f) The fact that the eigenvalues of R are complex means that the probabilities should *oscillate* as they decay to their equilibrium values $p_{\rm A} = p_{\rm B} = p_{\rm C} = \frac{1}{3}$. Can you see this in your simulations?