PHYSICS 200B : CLASSICAL MECHANICS
HOMEWORK SET #4

[1] Blasius' theorem says that the force per unit length of a body of constant cross-sectional profile Σ is given by

$$\overline{\mathcal{F}} = \mathcal{F}_x - i\mathcal{F}_y = \tfrac{i}{2}\,\rho \oint\limits_{\mathcal{C}} dz \left(\frac{dW}{dz}\right)^2 \quad ,$$

where $\mathcal{C} = \partial\Sigma$ is a closed curve which traces the boundary of Σ, and $W(z)$ is the complex potential.

Consider a 2D flow with stream function $\psi(x, y) = A(x - c)y$, where A and c are real constants. A circular cylinder of radius a is introduced into this flow, with its center at the origin. Find $W(z)$ for the resulting flow. Use Blasius' theorem to calculate the force per unit length exerted on the cylinder.

[2] Show that the Joukowski transformation $Z = z + a^2/z$ can be written in the form

$$\frac{Z - 2a}{Z + 2a} = \left(\frac{z - a}{z + a}\right)^2 \quad ,$$

so that

$$\mathsf{arg}(Z - 2a) - \mathsf{arg}(Z + 2a) = 2\Big\{\mathsf{arg}(z - a) - \mathsf{arg}(z + a)\Big\} \quad . \tag{1}$$

Consider the circle in the (x, y) plane which passes through $z = -a$ and a with its center at $z_0 = ia\,\text{ctn}\,\beta$. Show that the above transformation takes this circle into a circular arc between $Z = -2a$ and $Z = +2a$, with subtended angle 2β (see figure). Obtain an expression for the complex potential in the Z plane when the flow is uniform at speed V and parallel to the real axis. Show that the velocity will be finite at both the leading and tailing edges if $\Gamma - -4\pi V a\,\text{ctn}\,\beta$.

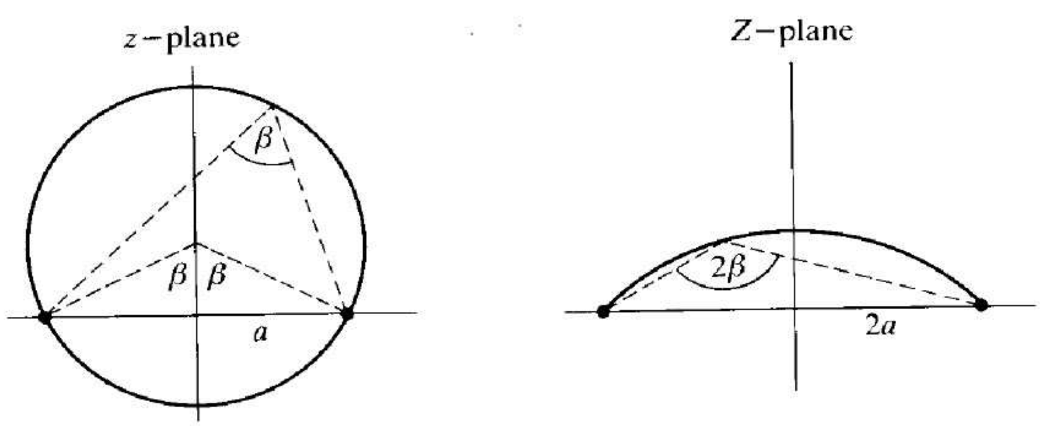

Figure 1: Geometry of the circle and its image in problem 2.

[3] Show that an array of N identical point vortices of circulation Γ, placed equally about a circle of radius a, will rotate at a constant angular frequency Ω. Find the value of Ω.

[4] Consider a large circular disk of radius R executing a prescribed angular motion $\theta(t)$. The disk is immersed in a fluid under conditions of constant pressure. Let the plane of the disk lie at $z = 0$. Assume that the fluid velocity takes the form

$$v_\phi(r, \phi, z, t) = r\,\Omega(z, t) \ , \tag{2}$$

with $v_r = v_z = 0$.

(a) Write down the Navier-Stokes equations for the fluid. Assume you can neglect the $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v}$ term. (Under what conditions is this true?) Show that you obtain the diffusion equation. What are the boundary conditions on the fluid motion?

(b) Our goal is next to find a complete solution to $\Omega(z, t)$ in terms of the function $\theta(t)$. To this end, we perform the following analysis. Define the spatial Laplace transform,

$$\check{\Omega}_{\mathsf{L}}(\kappa, t) \equiv \int\limits_0^\infty dz\, e^{-\kappa z}\, \Omega(z, t) \ . \tag{3}$$

You may assume in this problem that the fluid motion is symmetric about $z = 0$, *i.e.* $\Omega(z, t) = \Omega(-z, t)$, so we only have to consider the region $z \geq 0$. The inverse Laplace transform is

$$\Omega(z, t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{d\kappa}{2\pi i}\, e^{+\kappa z}\, \check{\Omega}_{\mathsf{L}}(\kappa, t) \tag{4}$$

where the contour lies to the left of any branch cut or singularity on the line $\mathrm{Im}\,(\kappa) = 0$. Later on we will see that we can take $c = 0$, so the contour lies along the axis $\mathrm{Re}\,(\kappa) = 0$. Show directly that

$$\big(\partial_t - \nu\kappa^2\big)\, \check{\Omega}_{\mathsf{L}}(\kappa, t) = F_\kappa(t) \ , \tag{5}$$

where the function $F_\kappa(t)$ on the RHS depends on $\Omega(0, t)$ and $\Omega'(0, t)$ (prime denotes differentiation with respect to z). Find $F_\kappa(t)$.

(c) Integrate the above first order equation from some arbitrary initial time $t = t_0$ to final time t and obtain $\Omega(z, t)$ in terms of the functions $\Omega(z, t_0)$, $\Omega(0, t)$, and $\Omega'(0, t)$. Show that the term involving $\Omega(z, t_0)$ is a transient which decays to zero in the limit $t_0 \to -\infty$. Dropping the transient, performing the inverse Laplace transform, and rotating the κ contour so that $\kappa = ik$, where k runs along the real axis, show that

$$\Omega(z, t) = -\nu \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikz} \int\limits_{-\infty}^{t} dt'\, e^{-\nu k^2 (t - t')} \Big[\Omega'(0, t') + ik\Omega(0, t')\Big] \ . \tag{6}$$

(d) Find the total torque on the disk $N(t)$. You will need to integrate $\boldsymbol{r} \times \boldsymbol{f}$ over the surface of the disk, using the viscous stress tensor of the fluid. Show that

$$N_{\text{fluid}}(t) = \pi\eta\, R^4\, \Omega'(0, t) \ , \tag{7}$$

where $\eta = \rho\,\nu$ is the shear viscosity.

(e) By going to Fourier space in frequency, the k integral can be done. Show that

$$\hat{\Omega}(z,\omega) = -\frac{i\,e^{ik_+ z}}{k_+ - k_-}\left\{\hat{\Omega}'(0,\omega) + ik_+\hat{\Omega}(0,\omega)\right\} , \tag{8}$$

where $k_\pm = \pm\, e^{i\pi/4}\,\sqrt{\omega/\nu}$. Thus, setting $z \to 0^+$, we obtain

$$\hat{\Omega}'(0,\omega) = -ik_-\,\hat{\Omega}(0,\omega) \ . \tag{9}$$

(f) Suppose the disk is suspended from a torsional fiber. Let the disk's moment of inertia be I and the restoring torque due to the fiber be $N_{\text{fiber}} = -K\theta$. Show that the equation for the oscillation frequency of the disk is

$$\omega^2 + e^{i\pi/4}\,\omega_\nu^{1/2}\,\omega^{3/2} - \omega_0^2 = 0 \ , \tag{10}$$

where $\omega_0 = (K/I)^{1/2}$, and

$$\omega_\nu = \frac{\pi^2\rho^2 R^8\,\nu}{I^2} \ . \tag{11}$$

Analyze this equation in the limits $\omega_0 \ll \omega_\nu$ and $\omega_0 \gg \omega_\nu$, and find the frequency of damped oscillations. *Hint:* The former case is easy – simply neglect the ω^2 term. For the latter case, perturb about the $\omega_\nu = 0$ solutions $\omega = \pm\omega_0$. Find the real and imaginary parts of the oscillation frequency ω in each case.

Note: There is an easier way to solve this problem, if we use some intuition. The diffusion equation $\Omega_t = \nu\Omega_{zz}$ and the boundary conditions are linear, which suggests we write our solution as

$$\Omega(z,t) = A(\omega)\,e^{-Q|z|}\,e^{-i\omega t} \ . \tag{12}$$

This is a solution to the diffusion equation if $\nu Q^2 = -i\omega$. Of the two roots for $Q(\omega)$, we need the one with the positive real part, so $Q = e^{-i\pi/4}\sqrt{\omega/\nu}$. Setting $z = 0$ and using $\dot{\Omega} = \theta$, we find $A(\omega) = -i\omega\,\hat{\theta}(\omega)$. The Fourier component of the viscous torque on the disk is then

$$\hat{N}_{\text{fluid}}(\omega) = \pi\rho\nu R^4 \cdot (-Q)(-i\omega)\,\hat{\theta}(\omega) \tag{13}$$

$$= e^{i\pi/4}\,\pi\rho R^4\nu^{1/2}\omega^{3/2}\,\hat{\theta}(\omega) \ , \tag{14}$$

which when plugged into the equation of motion for the disk yields the above equation for the oscillation frequency.