Lecture 20 (Dec. 9) : MAPS $\left(\vec{\varphi}_{n+1} = \hat{T}\vec{\varphi}_n\right)$

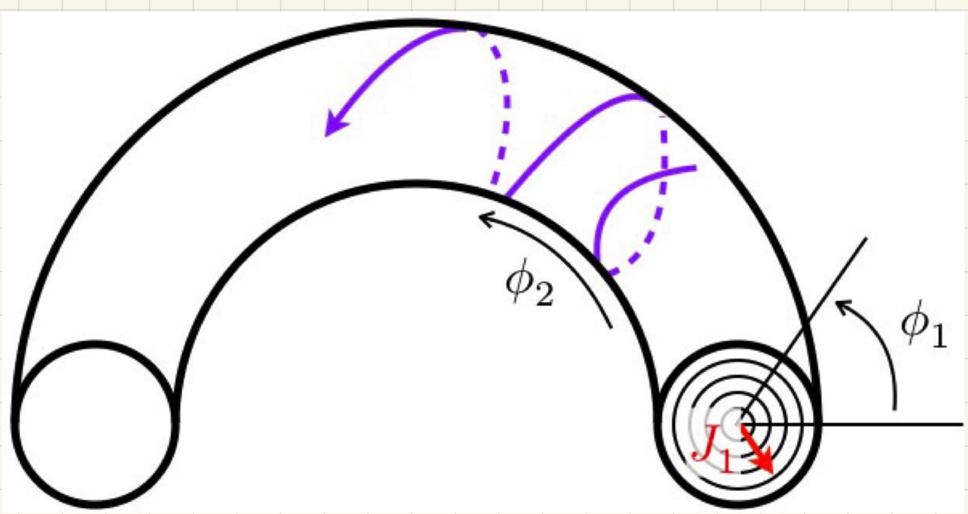
- **Motion on resonant tori**

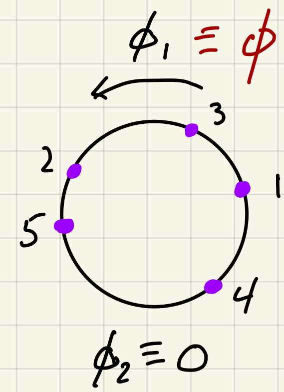
Consider the motion on a resonant torus in terms of the AAV:

$$\vec{\phi}(t) = \vec{\omega}(\vec{J})t + \vec{\phi}(0)$$

Resonance means that there exist some n-tuples $\vec{l} = \{l_1, \ldots, l_n\}$ for which $\vec{l}\cdot\vec{\omega} = 0$. If the motion is periodic, so that $\omega_j = k_j \omega_0$ with $k_j \in \mathbb{Z}$ for each $j \in \{1, \ldots, n\}$, then <u>all</u> of the frequencies are in resonance.

Let's consider the case $n=2$. Dynamics sketched below:





Since the energy E is fixed, we can regard $J_2 = J_2(J_1, E)$ and the motion as occurring in the 3-dim^{l} space (ϕ_1, ϕ_2, J_1). Suppose we plot the consecutive intersections of the system's motion with the two-dim^{l} subspace defined by fixing E and also ϕ_2 (say $\phi_2 \equiv 0$). Let's write $\phi \equiv \phi_1$ and $J \equiv J_1$, and define (ϕ_k, J_k) to be the values of (ϕ, J) at the k^{th} consecutive intersection of the system's motion with the subspace ($\phi_2 = 0$, E fixed). The 2d space (ϕ_2, J_2) is called the **surface of section**. Since $\dot{\phi}_2 = \omega_2$, we have

or E

$$\phi_{k+1} - \phi_k = \omega_1 \cdot \frac{2\pi}{\omega_2} \equiv 2\pi\alpha$$

$$\alpha(J) \equiv \frac{\omega_1(J)}{\omega_2(J)}$$

(E suppressed)

$\phi \equiv \phi_1$, $J \equiv J_1$

and therefore

$$\phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1})$$

$$J_{k+1} = J_k$$

"twist map"

Note that we've written here $\alpha(J_{n+1})$ in the first equation. [Since $J_{k+1} = J_k$, it doesn't matter since J never changes for these dynamics. But writing the equations this way is more convenient.] Note that $(\phi_n, J_n) \to (\phi_{n+1}, J_{n+1})$ is canonical:

$$\{\phi_{k+1}, J_{k+1}\}_{(\phi_k, J_k)} = \det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)}$$

$$= \frac{\partial\phi_{k+1}}{\partial\phi_k}\frac{\partial J_{k+1}}{\partial J_k} - \frac{\partial\phi_{k+1}}{\partial J_k}\frac{\partial J_{k+1}}{\partial\phi_k} = 1\cdot 1 - 0\cdot 0 = 1$$

Formally, we may write this map as

$$\vec{\varphi}_{k+1} = \hat{T}\vec{\varphi}_k$$

where $\vec{\varphi}_k = (\phi_k, J_k)$ and $\hat{T}$ is the map. Note that if

$\alpha = \frac{r}{s} \in \mathbb{Q}$, then $\hat{T}^s$ acts as the identity, leaving every point in the (ϕ, J) plane fixed.

For systems with n degrees of freedom, and with the surface of section fixed by (ϕ_n, J_n) or (ϕ_n, E), define $\vec{\varphi} \equiv (\phi_1, \dots, \phi_{n-1})$ and $\vec{J} \equiv (J_1, \dots, J_{n-1})$. Then with $\vec{\alpha} \equiv \left(\frac{\omega_1}{\omega_n}, \dots, \frac{\omega_{n-1}}{\omega_n}\right)$,

$$\vec{\varphi}_{k+1} = \vec{\varphi}_k + 2\pi\vec{\alpha}(\vec{J}_{k+1})$$
$$\vec{J}_{k+1} = \vec{J}_k$$

which is canonical. Note $\vec{\varphi}_k = (\varphi_{1,k}, \dots, \varphi_{n-1,k})$ where $\varphi_{j,k}$ is the value of φ_j the k^{th} time the motion passes through the SOS. We call this map the **twist map**.

Perturbed twist map: Now consider a Hamiltonian $H(\vec{\phi}, \vec{J}) = H_0(\vec{J}) + \epsilon H_1(\vec{\phi}, \vec{J})$. Again we will take $n=2$. We expect the resulting map on the SOS to be given by

$$\hat{T}_\epsilon \vec{\varphi}_k = \varphi_{k+1} : \begin{cases} \phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1}) + \epsilon f(\phi_k, J_{k+1}) + \dots \\ J_{k+1} = J_k + \epsilon g(\phi_k, J_{k+1}) + \dots \end{cases}$$

Is this map canonical? Let's check that $\det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)} = 1$:

$$d\phi_{k+1} = d\phi_k + 2\pi\alpha'(J_{k+1})dJ_{k+1} + \epsilon \frac{\partial f}{\partial \phi_k} d\phi_k + \epsilon \frac{\partial f}{\partial J_{k+1}} dJ_{k+1}$$
$$dJ_{k+1} = dJ_k + \epsilon \frac{\partial g}{\partial \phi_k} d\phi_k + \epsilon \frac{\partial g}{\partial J_{k+1}} dJ_{k+1}$$

Now bring $d\phi_{k+1}$ and dJ_{k+1} to the LHS of each eqn and bring $d\phi_k$ and dJ_k to the RHS. We obtain

$$\underbrace{\begin{pmatrix} 1 & -2\pi\alpha'(J_{k+1}) - \epsilon\dfrac{\partial f}{\partial J_{k+1}} \\ 0 & 1-\epsilon\dfrac{\partial g}{\partial J_{k+1}} \end{pmatrix}}_{A_{k+1}} \begin{pmatrix} d\phi_{k+1} \\ dJ_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1+\epsilon\dfrac{\partial f}{\partial \phi_k} & 0 \\ \epsilon\dfrac{\partial g}{\partial \phi_k} & 1 \end{pmatrix}}_{B_k} \begin{pmatrix} d\phi_k \\ dJ_k \end{pmatrix}$$

Thus

$$\det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)} = \frac{\det B_k}{\det A_{k+1}} = \frac{1+\epsilon\dfrac{\partial f}{\partial \phi_k}}{1-\epsilon\dfrac{\partial g}{\partial J_{k+1}}} \equiv 1$$

and we conclude the necessary condition is $\dfrac{\partial f}{\partial \phi_k} = \dfrac{\partial g}{\partial J_{k+1}}$.

This guarantees the map $\hat{T}_\epsilon$ is canonical.

If we restrict to $g = g(\phi)$, then we have $f = f(J)$.

We may then write $2\pi\alpha(J_{k+1}) + \epsilon f(J_{k+1}) \equiv 2\pi\alpha_\epsilon(J_{k+1})$.

(We'll drop the ϵ subscript on α.) Thus, our perturbed twist map is given by

$$\left.\begin{aligned} \phi_{k+1} &= \phi_k + 2\pi\alpha(J_{k+1}) \\ J_{k+1} &= J_k + \epsilon g(\phi_k) \end{aligned}\right\} \text{ canonical!}$$

For $\alpha(J) = J$ and $g(\phi) = -\sin\phi$, we obtain the **standard map**

$$\phi_{k+1} = \phi_k + 2\pi J_{k+1} \quad , \quad J_{k+1} = J_k - \epsilon \sin\phi_k$$

- Maps from time-dependent Hamiltonians

- **Parametric oscillator**, e.g. pendulum with time-dependent length $\ell(t)$: $\ddot{x} + \omega_0^2(t)\,x = 0$ with $\omega_0(t) = \sqrt{g/\ell(t)}$. This describes pumping a swing by periodically extending and withdrawing one's legs. We have

$$\underbrace{\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix}}_{\dot{\vec{\varphi}}(t)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}}_{A(t)} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{\vec{\varphi}(t)} \qquad (v = \dot{x})$$

The formal sol$^{\underline{n}}$ to $\dot{\vec{\varphi}}(t) = A(t)\vec{\varphi}(t)$ is

$$\vec{\varphi}(t) = T\exp\left\{\int_0^t dt'\,A(t')\right\}\vec{\varphi}(0)$$

where T is the **time ordering operator** which puts earlier times to the right. Thus

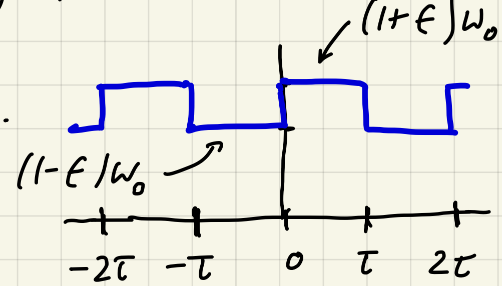
$$T\exp\left\{\int_0^t dt'\,A(t')\right\} = \lim_{N\to\infty}\big(1 + A(t_{N-1})\delta\big)\cdots\big(1 + A(0)\delta\big)$$

where $t_j = j\delta$ with $\delta \equiv t/N$. Note if $A(t)$ is time independent then

$$T\exp\left\{\int_0^t dt'\,A(t')\right\} = e^{At} = \lim_{N\to\infty}\left(1 + \frac{At}{N}\right)^N$$

There are no general methods for analytically evaluating time-ordered exponentials as we have here. But one tractable case is where the matrix $A(t)$ oscillates as a square wave:

$$\omega(t) = \begin{cases} (1+\epsilon)\,\omega_0 & \text{if } 2j\tau \leq t < (2j+1)\tau \\ (1-\epsilon)\,\omega_0 & \text{if } (2j+1)\tau \leq t < (2j+2)\tau \end{cases} \qquad (\text{for } j \in \mathbb{Z})$$



The period is 2τ. Define $\vec{\varphi}_n = \vec{\varphi}(t = 2n\tau)$. Then we have

$$\vec{\varphi}_{n+1} = e^{A_-\tau}\, e^{A_+\tau}\, \vec{\varphi}_n$$

NB: $e^{A_-\tau} e^{A_+\tau} \neq e^{(A_- + A_+)\tau}$

with

$$A_\pm = \begin{pmatrix} 0 & 1 \\ -\omega_\pm^2 & 0 \end{pmatrix} \quad , \qquad \omega_\pm \equiv (1 \pm \epsilon)\,\omega_0$$

Note that $A_\pm^2 = -\omega_\pm^2 \mathbb{1}$ and that

$$\begin{aligned} \mathcal{U}_\pm \equiv e^{A_\pm\tau} &= \mathbb{1} + A_\pm \tau + \frac{1}{2!} A_\pm^2 \tau^2 + \frac{1}{3!} A_\pm^3 \tau^3 + \dots \\ &= \left(1 - \frac{1}{2!}\omega_\pm^2 \tau^2 + \frac{1}{4!}\omega_\pm^4 \tau^4 + \dots\right)\mathbb{1} \\ &\quad + \left(\tau - \frac{1}{3!}\omega_\pm^2 \tau^3 + \frac{1}{5!}\omega_\pm^4 \tau^5 - \dots\right) A_\pm \\ &= \cos(\omega_\pm \tau)\,\mathbb{1} + \omega_\pm^{-1} \sin(\omega_\pm \tau)\, A_\pm \\ &= \begin{pmatrix} \cos(\omega_\pm \tau) & \omega_\pm^{-1}\sin(\omega_\pm \tau) \\ -\omega_\pm \sin(\omega_\pm \tau) & \cos(\omega_\pm \tau) \end{pmatrix} = e^{A_\pm \tau} \end{aligned}$$

M symplectic $\Rightarrow$
$M^t J M = J$
$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$

Note also that $\det \mathcal{U}_\pm = 1$, since $\mathcal{U}_\pm$ is simply Hamiltonian evolution over half a period, and it must be canonical.

Now we need

$$\mathcal{U} = \hat{T} \exp\left\{\int_0^{2\tau} dt\, A(t)\right\} = \mathcal{U}_- \mathcal{U}_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(real, not symmetric)

$$a = \cos(\omega_-\tau)\cos(\omega_+\tau) - \omega_-^{-1}\omega_+ \sin(\omega_-\tau)\sin(\omega_+\tau)$$

$$b = \omega_+^{-1}\cos(\omega_-\tau)\sin(\omega_+\tau) + \omega_-^{-1}\sin(\omega_-\tau)\cos(\omega_+\tau)$$

$$c = -\omega_+\cos(\omega_-\tau)\sin(\omega_+\tau) - \omega_-\sin(\omega_-\tau)\cos(\omega_+\tau)$$

$$d = \cos(\omega_-\tau)\cos(\omega_+\tau) - \omega_+^{-1}\omega_-\sin(\omega_-\tau)\sin(\omega_+\tau)$$

It follows from $\mathcal{U} = \mathcal{U}_-\mathcal{U}_+$ that $\mathcal{U}$ is also canonical (i.e. $\vec{\varphi}_{n+1} = \mathcal{U}\vec{\varphi}_n$ is a canonical transformation).

The eigenvalues $\lambda_\pm$ of $\mathcal{U}$ thus satisfy $\lambda_+\lambda_- = 1$. For a 2×2 matrix $\mathcal{U} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, the characteristic polynomial is

$$P(\lambda) = \det(\lambda \mathbb{1} - \mathcal{U}) = \lambda^2 - T\lambda + \Delta$$

where $T = \mathrm{tr}\,\mathcal{U} = a+d$ and $\Delta = \det\mathcal{U} = ad - bc$. The eigenvalues are then

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2 - 4\Delta}$$

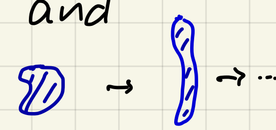
But in our case $\mathcal{U}$ is special, and $\det\mathcal{U} = 1$, so

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2-4} \;=\; \frac{T}{2} \pm i\sqrt{1-\left(\frac{T}{2}\right)^2}$$

We therefore have:

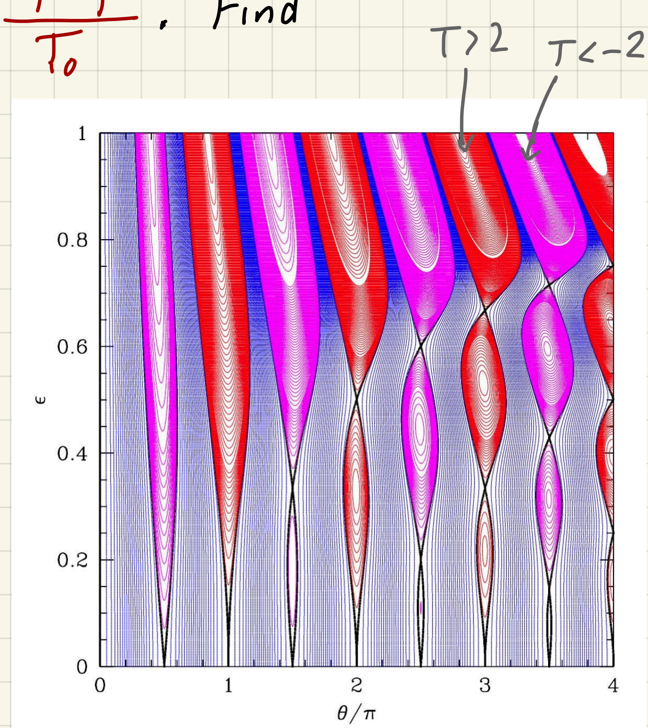
$$|T| < 2 \;:\; \lambda_+ = \lambda_-^* = e^{i\delta} \quad \text{with } \delta = \cos^{-1}\!\left(\tfrac{1}{2}T\right)$$

$$|T| > 2 \;:\; \lambda_+ = \lambda_-^{-1} = e^{\mu}\,\mathrm{sgn}(T) \quad \text{with } \mu = \cosh^{-1}\!\left(\tfrac{1}{2}|T|\right)$$

Note $\lambda_+\lambda_- = \det \mathcal{U} = 1$ always. Thus, for $|T| < 2$, the motion is bounded, but for $|T| > 2$ we have that $|\vec{\varphi}|$ increases exponentially with time, even though phase space volumes are preserved by the dynamics. I.e. we have exponential stretching along the eigenvector $\vec{\psi}_+$ and exponential squeezing along the eigenvector $\vec{\psi}_-$. 

Let's set $\theta = \omega_0 \tau = 2\pi\tau/T_0$ where T_0 is the natural oscillation period when $\epsilon = 0$. Since the period of the pumping is $T_{pump} = 2\tau$, we have $\dfrac{\theta}{\pi} = \dfrac{T_{pump}}{T_0}$. Find

$$\mathrm{Tr}\,\mathcal{U} = \frac{2\cos(2\theta) - 2\epsilon^2\cos(2\epsilon\theta)}{1-\epsilon^2}$$



$$T = +2 : \;\; \theta = n\pi + \delta\,, \;\; \epsilon = \pm\left|\frac{\delta}{n\pi}\right|^{1/2}$$

$$T = -2 : \;\; \theta = (n+\tfrac{1}{2})\pi + \delta\,, \;\; \epsilon = \pm\,\delta$$

The phase diagram in (θ, ϵ) space is shown at the right.

Kicked dynamics: Let $H(t) = T(p) + V(q)\,K(t)$, where

$$K(t) = \tau \sum_{-\infty}^{\infty} \delta(t - n\tau)$$

"Dirac comb"

As $\tau \to 0$, $K(t) \to 1$ (constant).

Equations of motion:

$$\dot{q} = T'(p) \quad , \quad \dot{p} = -V'(q)\,K(t)$$

Define $q_n \equiv q(t = n\tau^+)$ and $p_n = p(t = n\tau^+)$ and integrate from $t = n\tau^+$ to $t = (n+1)\tau^+$:

$$q_{n+1} = q_n + \tau\,T'(p_n)$$
$$p_{n+1} = p_n - \tau\,V'(q_{n+1})$$

This is our map $\vec{\varphi}_{n+1} = \hat{T}\vec{\varphi}_n$. Note that it is q_{n+1} which appears as the argument of V' in the second equation. This is crucial in order that $\hat{T}$ be canonical:

$$dq_{n+1} = dq_n + \tau\,T''(p_n)\,dp_n$$
$$dp_{n+1} = dp_n - \tau\,V''(q_{n+1})\,dq_{n+1}$$

$$\begin{pmatrix} 1 & 0 \\ \tau V''(q_{n+1}) & 1 \end{pmatrix} \begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \tau T''(p_n) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dq_n \\ dp_n \end{pmatrix}$$

$$\begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \tau T''(p_n) \\ -\tau V''(q_{n+1}) & 1 - \tau^2 T''(p_n) V''(q_{n+1}) \end{pmatrix} \begin{pmatrix} dq_n \\ dp_n \end{pmatrix}$$

and thus

$$\det \frac{\partial(q_n, p_n)}{\partial(q_{n+1}, p_{n+1})} = 1$$

The **standard map** is obtained from

$$H(t) = \frac{L^2}{2I} - V\cos\phi \, K(t)$$

resulting in

$$\begin{aligned} \phi_{n+1} &= \phi_n + \frac{\tau}{I} L_n \\ L_{n+1} &= L_n - \tau V \sin\phi_{n+1} \end{aligned}$$

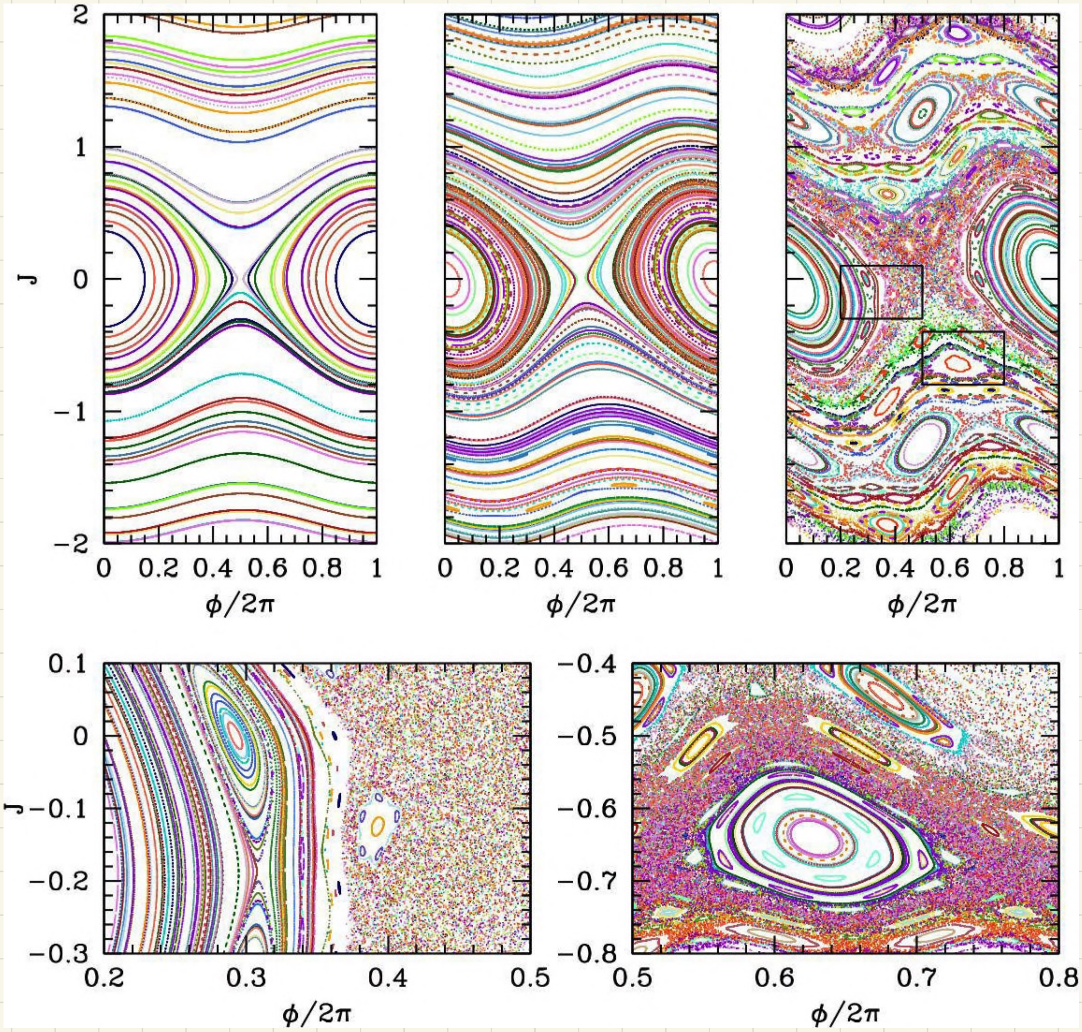
Defining $J_n \equiv L_n / \sqrt{2\pi I V}$ and $\epsilon \equiv \tau \sqrt{V/2\pi I}$ we arrive at

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\epsilon J_n \\ J_{n+1} &= J_n - \epsilon \sin\phi_{n+1} \end{aligned}$$

The phase space (ϕ, J) is thus a cylinder. As $\epsilon \to 0$,

$$\left.\begin{aligned} \frac{\phi_{n+1} - \phi_n}{\epsilon} &\to \frac{d\phi}{ds} = 2\pi J \\ \frac{J_{n+1} - J_n}{\epsilon} &\to \frac{dJ}{ds} = -\sin\phi \end{aligned}\right\} \Rightarrow \begin{array}{l} E = \pi J^2 - \cos\phi \\ \text{is preserved} \\ \text{pendulum!} \end{array}$$

This is because $\epsilon \to 0$ means $\tau \to 0$ hence $K(t) \to 1$, which is the simple pendulum. There is a separatrix at $E = 1$, along which $J(\phi) = \pm \frac{2}{\pi} |\cos(\phi/2)|$.



Top: $\epsilon = 0.01$ (left), $\epsilon = 0.2$ (center), $\epsilon = 0.4$ (right)
Bottom: details from $\epsilon = 0.4$ (upper right)

Another example is the **kicked Harper map**, when

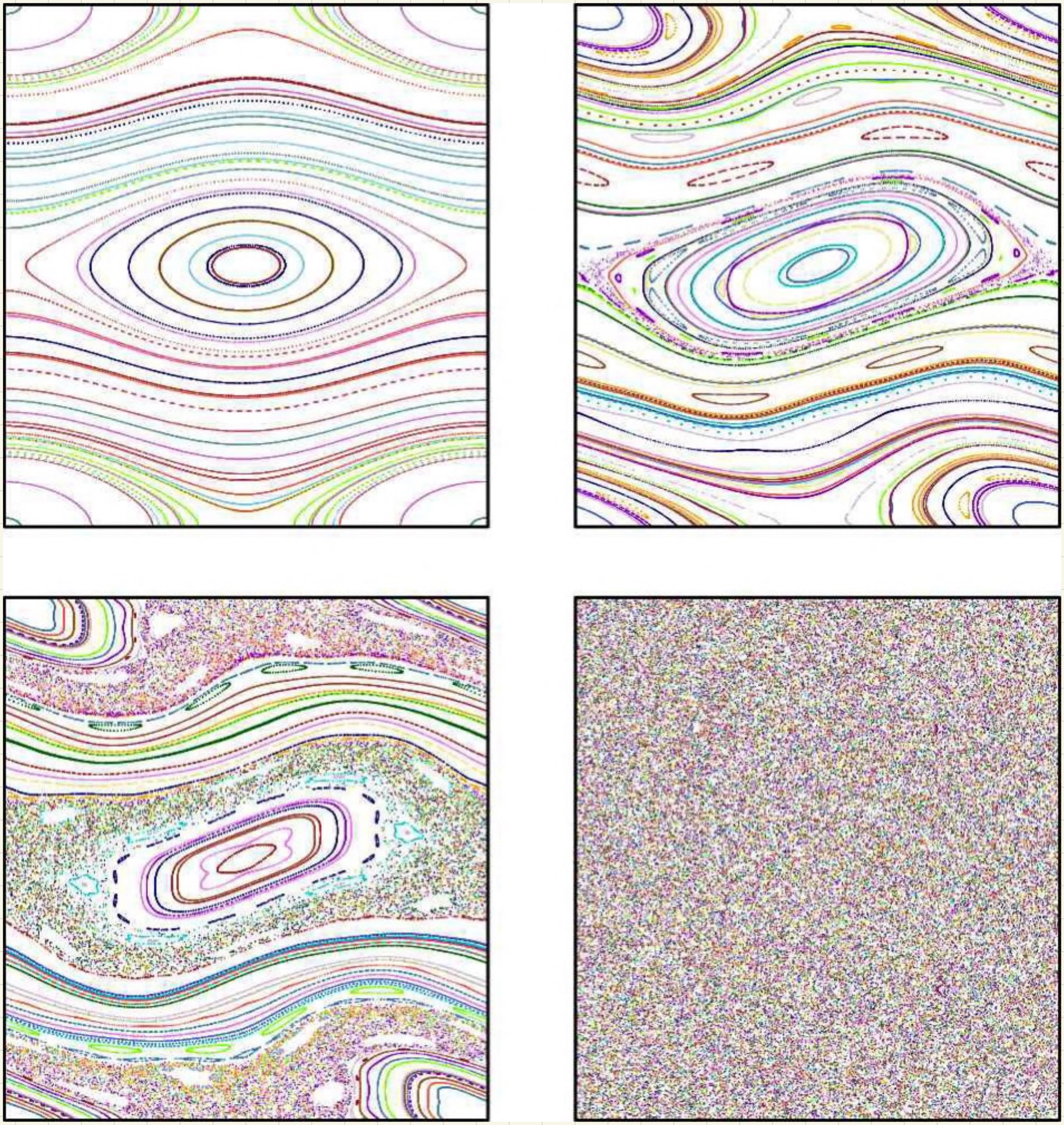
$$H(t) = -V_1 \cos\left(\frac{2\pi p}{P}\right) - V_2 \cos\left(\frac{2\pi q}{Q}\right) K(t)$$

This generates the map

$$x_{n+1} = x_n + \alpha \epsilon \sin(2\pi y_n) \qquad x \equiv q/Q \qquad \alpha = \sqrt{V_1/V_2}$$

$$y_{n+1} = y_n - \alpha^{-1} \epsilon \sin(2\pi x_{n+1}) \qquad y \equiv p/P \qquad \epsilon = \frac{2\pi\tau\sqrt{V_1 V_2}}{PQ}$$

on the torus $T^2 = [0,1] \times [0,1]$ with $x = 0,1$ identified and $y = 0,1$ identified.



Kicked Harper map with $\alpha = 2$ and $\epsilon = 0.01$ (UL), $\epsilon = 0.125$ (UR), $\epsilon = 0.2$ (LL), and $\epsilon = 5.0$ (LR).

Note PSF says $K(t) = \tau \sum_{-\infty}^{\infty} \delta(t - n\tau) = \sum_{-\infty}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right)$

and a kicked Hamiltonian may be written

$$H(J, \phi, t) = \underbrace{H_0(J) + V(\phi)}_{\text{integrable}} + \underbrace{2V(\phi) \sum_{m=1}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right)}_{\text{resonances}}$$

Local Stability and Lyapunov Exponents

Consider a map $\hat{T}$ on a phase space of dimension $n = 2N$. What is the fate of two nearly separated initial conditions $\vec{\xi}_0$ and $\vec{\xi}_0 + d\vec{\xi}$ under iterations of $\hat{T}$? First iteration:

$$\vec{\xi}_0 \to \vec{\xi}_1 = \hat{T}\vec{\xi}_0$$
$$\vec{\xi}_0 + d\vec{\xi} \to \hat{T}(\vec{\xi}_0 + d\vec{\xi}) = \vec{\xi}_1 + M(\vec{\xi}_0)\, d\vec{\xi} + \dots$$

where

$$M_{ij}(\vec{\xi}) = \frac{\partial(\hat{T}\vec{\xi})_i}{\partial \xi_j} \quad \text{an } n\times n \text{ matrix}$$

is the linearization of $\hat{T}$ at $\vec{\xi}$. Next iteration

$$\vec{\xi}_0 \longrightarrow \vec{\xi}_1 = \hat{T}\vec{\xi}_0 \longrightarrow \vec{\xi}_2 = \hat{T}\vec{\xi}_1 = \hat{T}^2\vec{\xi}_0$$
$$\vec{\xi}_0 + d\vec{\xi} \longrightarrow \vec{\xi}_1 + M(\vec{\xi}_0)\, d\vec{\xi} \longrightarrow \vec{\xi}_2 + M(\vec{\xi}_1) M(\vec{\xi}_0)\, d\vec{\xi}$$

Thus, after k iterations,

$$\vec{\xi}_0 \longrightarrow \vec{\xi}_k \equiv \hat{T}^k \vec{\xi}_0$$
$$\vec{\xi}_0 + d\vec{\xi} \longrightarrow \vec{\xi}_k + \underbrace{M(\vec{\xi}_{k-1}) M(\vec{\xi}_{k-2}) \cdots M(\vec{\xi}_0)}_{\text{product of } k \text{ matrices } R^{(k)}(\vec{\xi}_0)}\, d\vec{\xi}$$

We define the linear operator (matrix) $R^{(k)}(\vec{\xi})$ as

$$R^{(k)}(\vec{\xi}) = M(\hat{T}^{k-1}\vec{\xi})\, M(\hat{T}^{k-2}\vec{\xi}) \cdots M(\hat{T}\vec{\xi})\, M(\vec{\xi})$$

Thus,

$$R^{(k)}_{ij}(\vec{\xi}) = \frac{\partial(\hat{T}^k\vec{\xi})_i}{\partial \xi_j} \qquad \langle L^\alpha | R^\beta \rangle = \delta^{\alpha\beta}$$

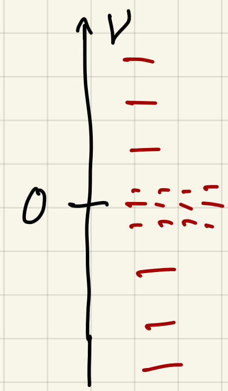
Since $\hat{T}$ is presumed canonical, at each stage the matrix $M(\vec{\xi}_j) \in Sp(2N)$, i.e. $M^t J M = J$ where $J = \begin{pmatrix} 0 & \mathbb{1}_{N\times N} \\ -\mathbb{1}_{N\times N} & 0 \end{pmatrix}$. As the product of symplectic matrices is itself symplectic, $R^{(k)}(\vec{\xi}) \in Sp(2N)$ for all $k, \vec{\xi}$. Note $J^2 = -\mathbb{1}$ so $M^{-1} = -JM^tJ$, and we have

$$\begin{aligned}
P(\lambda) &= \det(\lambda - R) = \det(\lambda^* - R) = P(\lambda^*) \\
&= \det(R^{-1} - \lambda^{-1}) \cdot \det R \cdot \lambda^n \\
&= \det(-JR^tJ - \lambda^{-1}) \cdot \det R \cdot \lambda^n \\
&= \det(\lambda^{-1} - R^t) \cdot \det R \cdot (-\lambda)^n \qquad ; \quad (-1)^n = (-1)^{2N} = 1 \\
&= \lambda^n \det R \cdot P(\lambda^{-1})
\end{aligned}$$

Thus, $P(\lambda) = 0 \Rightarrow P(\lambda^{-1}) = P(\lambda^*) = P(\lambda^{-1*}) = 0$, and the eigenvalues of any symplectic matrix come as

- either • unimodular pairs $(e^{i\delta}, e^{-i\delta})$, $\delta \in [0, 2\pi)$
- or • real pairs (λ, λ^{-1}), $\lambda \in \mathbb{R}$
- or • complex quartets $(\lambda, \lambda^{-1}, \lambda^*, \lambda^{*-1})$

One defines the **Lyapunov exponents**



$$\nu_j(\vec{\xi}) = \lim_{k\to\infty} \frac{1}{k} \ln |\lambda_j^{(k)}(\vec{\xi})|$$

where $\lambda_j^{(k)}(\vec{\xi})$ is the j^{th} eigenvalue of $R^{(k)}(\vec{\xi})$, ordered such that $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{2N}$. Note that $\nu_j + \nu_{2N+1-j} = 0$ and so there is a sum rule $\sum_{j=1}^{2N} \nu_j = 0$.

Note: $\nu_j < 0 \Rightarrow$ exponential squeezing, $\nu_j > 0 \Rightarrow$ exponential stretching

As an example, consider the Arnol'd cat map,

$$q_{t+1} = (r+1)q_t + p_t \mod 1 \qquad , \quad r \in \mathbb{Z}$$
$$p_{t+1} = r q_t + p_t \mod 1$$

Then

$$M = \frac{\partial(q_{t+1}, p_{t+1})}{\partial(q_t, p_t)} = \begin{pmatrix} r+1 & 1 \\ r & 1 \end{pmatrix} . \qquad \det M = 1$$

$$M^{-1} = \begin{pmatrix} 1 & -1 \\ -r & r+1 \end{pmatrix} \quad ; \quad M^t J M = J$$

The eigenvalues are $\lambda_\pm = 1 + \frac{r}{2} \pm \sqrt{r + \frac{r^2}{4}}$.

$r < -4$: $\lambda_- < -1 < \lambda_+ < 0$

$r = -4$: $\lambda_\pm = e^{\pm i\pi} = -1$

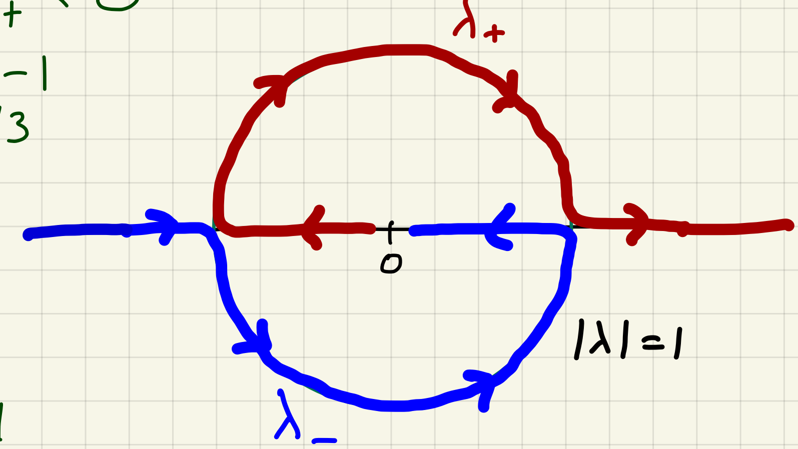
$r = -3$: $\lambda_\pm = e^{\pm 2\pi i/3}$

$r = -2$: $\lambda_\pm = e^{\pm i\pi/2}$

$r = -1$: $\lambda_\pm = e^{\pm i\pi/3}$

$r = 0$: $\lambda_\pm = e^{\pm i0} = 1$

$r > 0$: $0 < \lambda_- < 1 < \lambda_+$



The Lyapunov exponents are $\nu_\pm = \ln|\lambda_\pm|$, $\nu_+ + \nu_- = 0$.

Kolmogorov-Sinai entropy

Let Γ be our phase space, restricted to constant total energy E for Hamiltonian systems. Let $\{\Delta_j\}$ be a partition of disjoint sets whose union is Γ : $\bigcup_j \Delta_j = \Gamma$

$x \in [n, n+1) \Rightarrow n \leq x < n+1$

It is simplest to think of each Δ_j as a little hypercube. Stacking the hypercubes results in Γ. Now consider a map $\hat{T} : \Gamma \to \Gamma$ and consider the application of $\hat{T}$ to Δ_j. Then define $\Delta_{jk} \equiv \Delta_j \cap \hat{T}^{-1} \Delta_k$. Any point $\vec{\xi} \in \Delta_{jk}$ then satisfies $\vec{\xi} \in \Delta_j$ and $\hat{T}\vec{\xi} \in \Delta_k$. If $\sum_j \mu(\Delta_j) = \mu(\Gamma) = 1$, where $\mu(\Omega)$ is the measure of the set Ω, then we must have $\sum_{j,k} \mu(\Delta_{jk}) = 1$ because $\bigcup_k \Delta_{jk} = \Delta_j$. Now iterate once more, defining $\Delta_{jkl} \equiv \Delta_j \cap \hat{T}^{-1} \Delta_k \cap \hat{T}^{-2} \Delta_l$. Thus if $\vec{\xi} \in \Delta_{jkl}$, we have $\vec{\xi} \in \Delta_j$, $\hat{T}\vec{\xi} \in \Delta_k$, and $\hat{T}^2 \vec{\xi} \in \Delta_l$. The **entropy** of a distribution $\{p_a\}$, with $p_a \geq 0 \ \forall a$ and $\sum_a p_a = 1$, is defined to be $S(p) = -\sum_a p_a \log p_a$. We now define

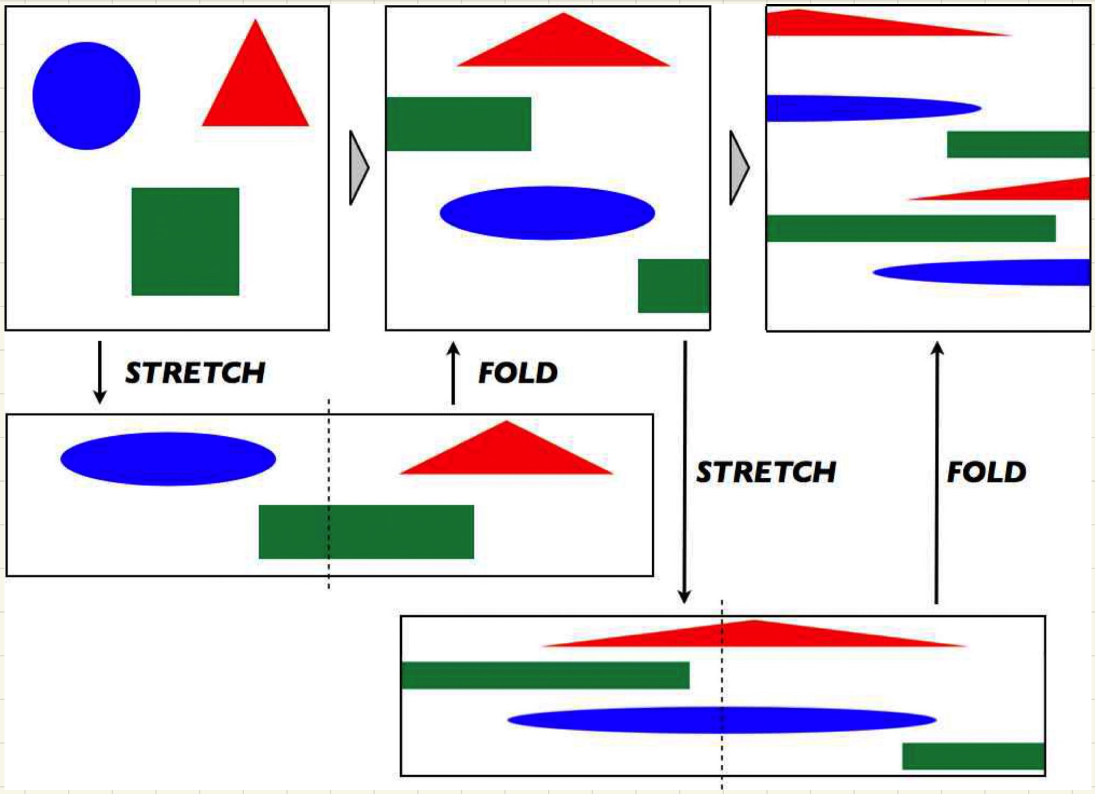
$$\Delta \equiv \{\Delta_j\}$$

$$S_L(\Delta) \equiv -\sum_{j_1} \cdots \sum_{j_L} \mu(\Delta_{j_1 \cdots j_L}) \log \mu(\Delta_{j_1 \cdots j_L})$$

This depends both on the initial partition $\{\Delta_j\}$ as well as the iteration number L. The **Kolmogorov-Sinai entropy** of the map $\hat{T}$ on the phase space Γ is then defined to be

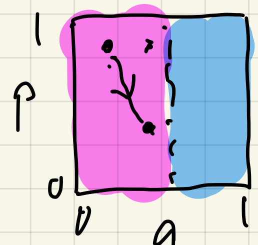
$$h_{KS} \equiv \sup_{\Delta} \lim_{L \to \infty} \frac{1}{L} S_L(\Delta)$$

where $\sup_\Delta$ indicates the supremum (maximum value) over over all possible partitions $\{\Delta_j\}$.



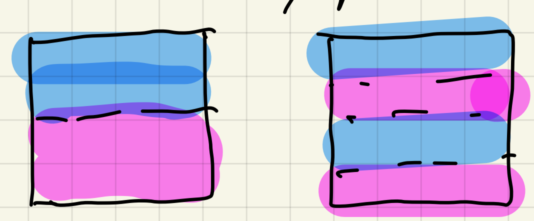
Consider the baker's transformation $\hat{T} : T^2 \to T^2$:

$$\hat{T}(q,p) = \begin{cases} (2q, \tfrac{1}{2}p) & \text{if } 0 \le q < \tfrac{1}{2} \\ (2q-1, \tfrac{1}{2}p + \tfrac{1}{2}) & \text{if } \tfrac{1}{2} \le q < 1 \end{cases}$$



Then it is straight forward to show that $h_{KS} = \ln 2$.

For a simple translation, $\hat{T}(q,p) = (q+\alpha, p+\beta) \bmod \mathbb{Z}^2$, then $h_{KS} = 0$. The KS entropy is related to the Lyapunov exponents according to



$$h_{KS} = \sum_j \nu_j \, \Theta(\nu_j)$$

i.e. a sum over positive Lyapunov exponents. If the $\nu_j(\vec{\xi})$ vary in space, then

$$h_{KS} = \int_\Gamma d\mu(\vec{\xi}) \sum_j \nu_j(\vec{\xi}) \, \Theta(\nu_j(\vec{\xi}))$$

"Pesin's entropy formula"

Poincaré-Birkhoff Theorem

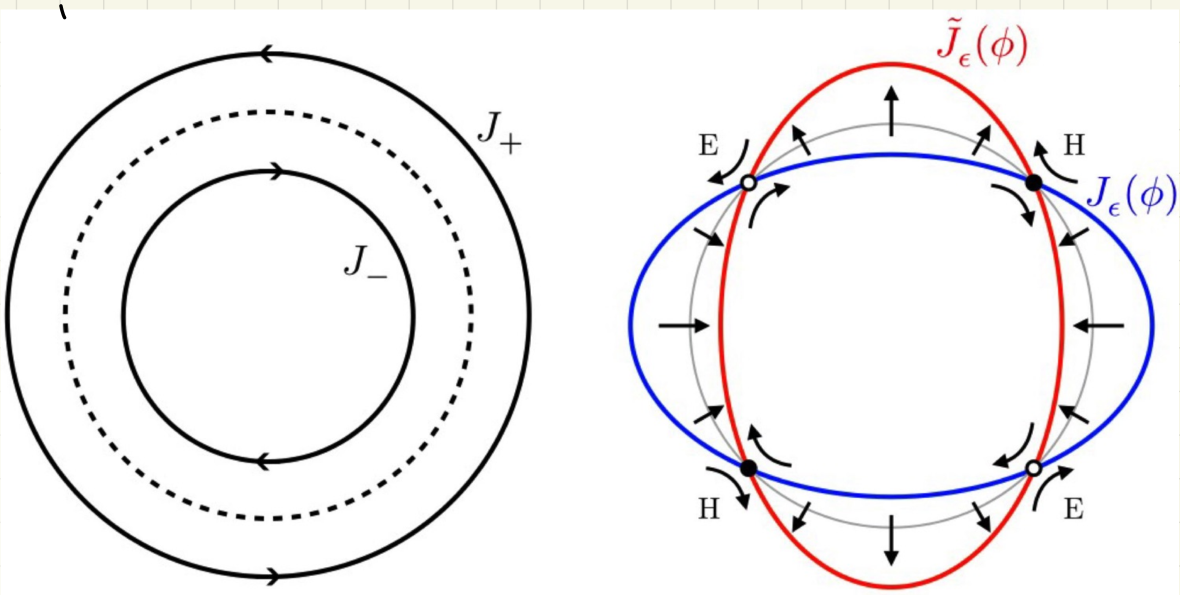
Back to our perturbed twist map, $\hat{T}_\epsilon$: $\tilde{\alpha}(J) \equiv \alpha(J) + \frac{\epsilon}{2\pi} f(J)$

$$\phi_{n+1} = \phi_n + 2\pi\alpha(J_{n+1}) + \epsilon f(\phi_n, J_{n+1}) = \phi_n + 2\pi\tilde{\alpha}(J_{n+1})$$
$$J_{n+1} = J_n + \epsilon g(\phi_n, J_{n+1}) \qquad = J_n + \epsilon g(\phi_n)$$

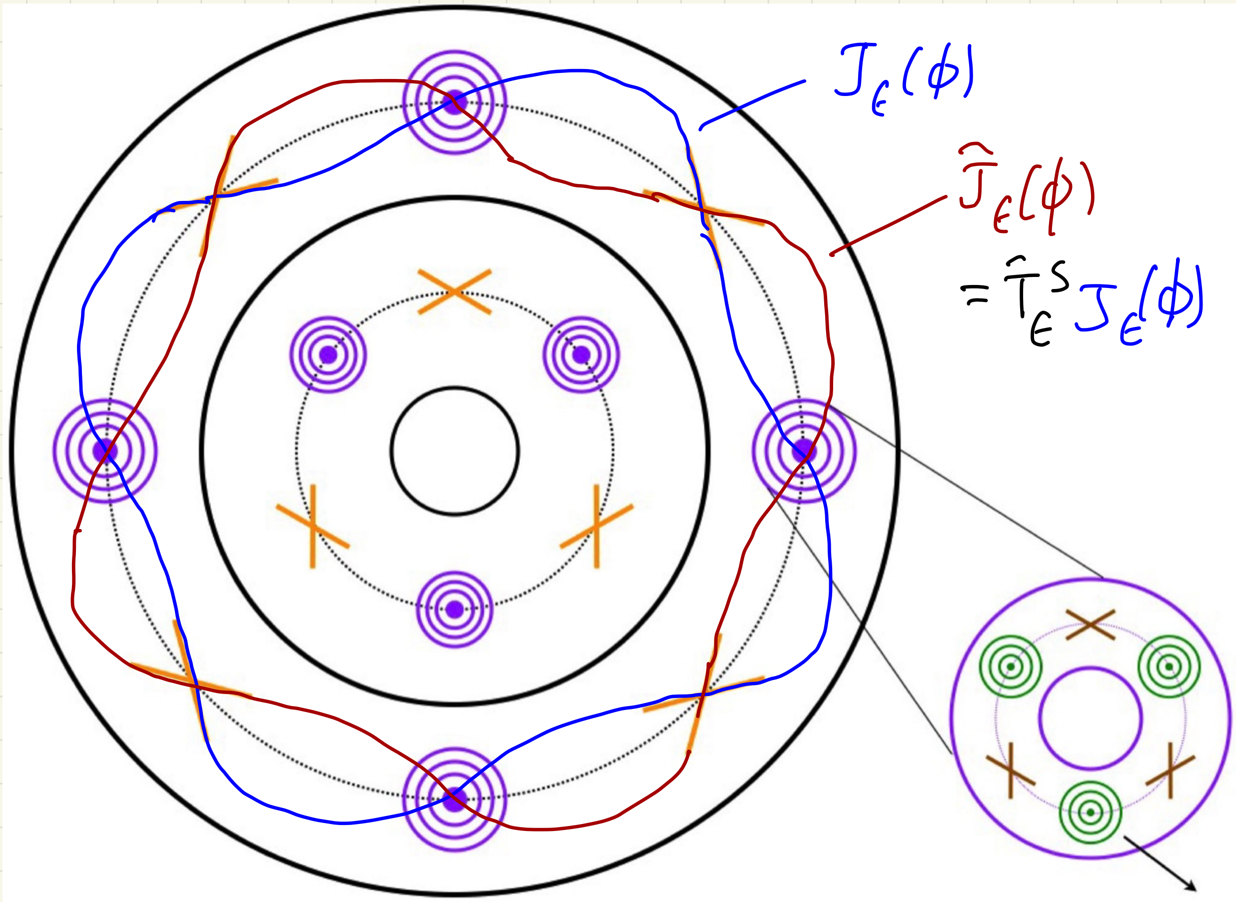
with

$$\frac{\partial f}{\partial \phi_n} + \frac{\partial g}{\partial J_{n+1}} = 0 \quad \Rightarrow \quad \hat{T}_\epsilon \text{ canonical}$$

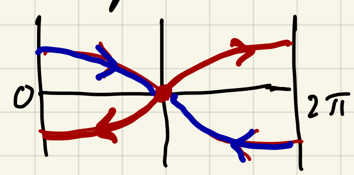
For $\epsilon = 0$, the map $\hat{T}_0$ leaves J invariant, and thus maps circles to circles. If $\alpha(J) \notin \mathbb{Q}$, the images of the iterated map $\hat{T}_0$ become dense on the circle. Suppose $\alpha(J) = \frac{r}{s} \in \mathbb{Q}$, and wolog assume $\alpha'(J) > 0$, so that on circles $J_\pm = J \pm \Delta J$ we have $\alpha(J_+) > r/s$ and $\alpha(J_-) < r/s$. Under $\hat{T}_0^{\,s}$, all points on the circle $C = C(J)$ are fixed. The circle $C_+ = C(J_+)$ rotates slightly counterclockwise while $C_- = C(J_-)$ rotates slightly clockwise. Now consider the action of $\hat{T}_\epsilon^{\,s}$, assuming that $\epsilon \ll \Delta J/J$. Acting on C_+, the result is still a net counterclockwise shift plus a small radial component of $\mathcal{O}(\epsilon)$. Similarly, C_- continues to rotate clockwise plus an $\mathcal{O}(\epsilon)$ radial component. By the Intermediate Value Theorem, for each value of ϕ there is some point $J = J_\epsilon(\phi)$ where the angular shift vanishes. Thus, along the curve $J_\epsilon(\phi)$ the action of $\hat{T}_\epsilon^s$ is **purely radial**. Next consider the curve $\tilde{J}_\epsilon(\phi) = \hat{T}_\epsilon^s J_\epsilon(\phi)$. Since $\hat{T}_\epsilon^s$ is volume-preserving, these curves must intersect at an even number of points.



The situation is depicted in the above figure. The intersections of $J_\epsilon(\phi)$ and $\tilde{J}_\epsilon(\phi)$ are thus **fixed points** of the map $\hat{T}_\epsilon^s$. We furthermore see that the intersection $J_\epsilon(\phi) \cap \tilde{J}_\epsilon(\phi)$ consists of an alternating sequence of elliptic and hyperbolic fixed points. This is the content of the PBT: a small perturbation of a resonant torus with $\alpha(J) = r/s$ results in an equal number of elliptic and hyperbolic fixed points for $\hat{T}_\epsilon^s$. Since $\hat{T}_\epsilon$ has period s acting on these fixed points, the number of EFPs and HFPs must be equal and a multiple of s. **In the vicinity of each EFP, this structure repeats** (see the figure below).

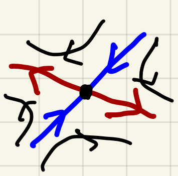


Self-similar structures in the iterated twist map.



Stable and unstable manifolds

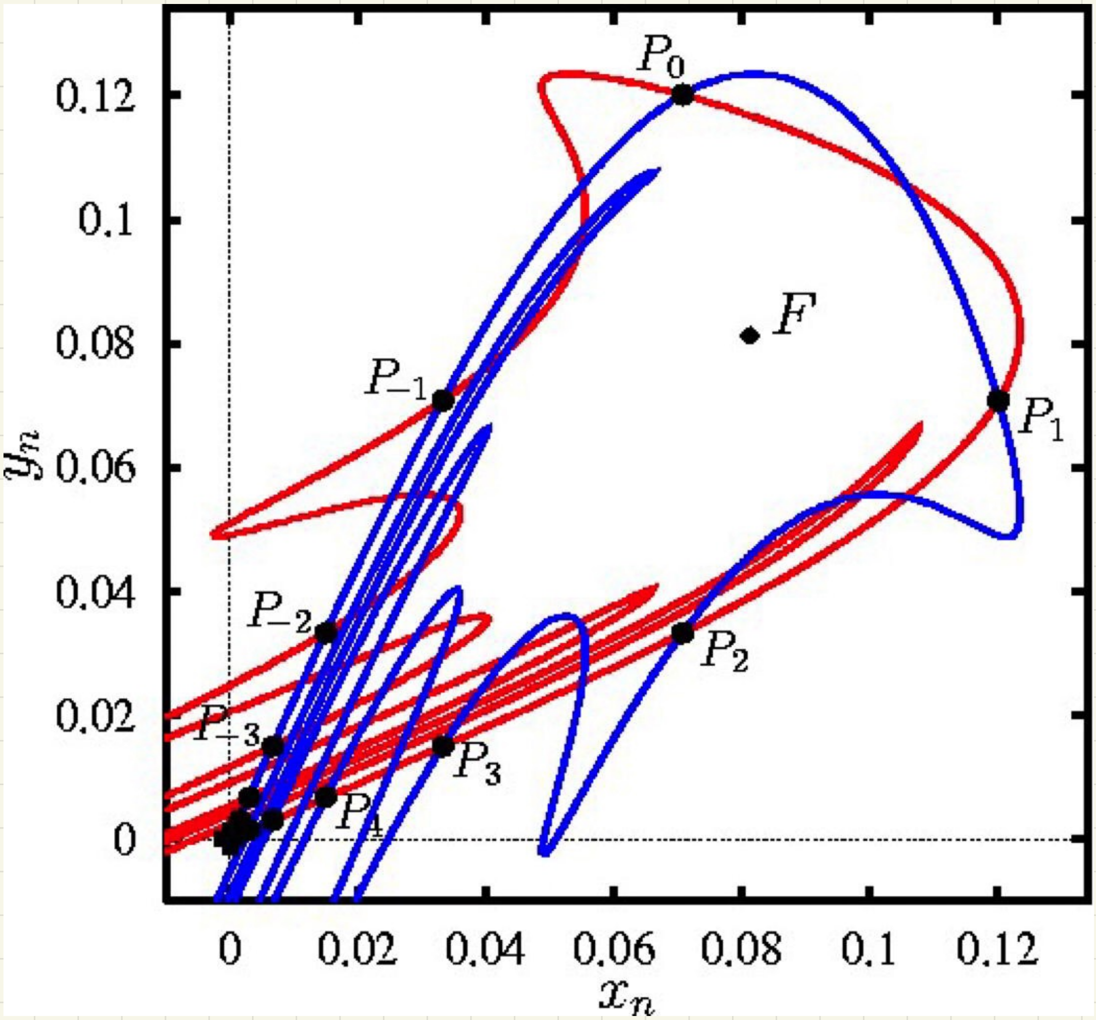
Emanating from each HFP are stable and unstable manifolds:



$$\vec{\varphi} \in \Sigma^{s}(\vec{\varphi}^{*}) \Rightarrow \lim_{n\to\infty} \hat{T}_\epsilon^{ns} \vec{\varphi} = \vec{\varphi}^{*} \quad (\text{flows } \textbf{to } \vec{\varphi}^{*})$$

$$\vec{\varphi} \in \Sigma^{u}(\vec{\varphi}^{*}) \Rightarrow \lim_{n\to\infty} \hat{T}_\epsilon^{-ns} \vec{\varphi} = \vec{\varphi}^{*} \quad (\text{flows } \textbf{from } \vec{\varphi}^{*})$$

Note $\Sigma^{s}(\vec{\varphi}_i^{*}) \cap \Sigma^{s}(\vec{\varphi}_j^{*}) = \phi$ and $\Sigma^{u}(\vec{\varphi}_i^{*}) \cap \Sigma^{u}(\vec{\varphi}_j^{*}) = \phi$ for $i \neq j$ (no s/s or u/u intersections). However, $\Sigma^{s}(\vec{\varphi}_i^{*})$ and $\Sigma^{u}(\vec{\varphi}_j^{*})$ **can** intersect. For $i = j$, this is called a **homoclinic point**. (On its way from $\vec{\varphi}_j^{*}$ to $\vec{\varphi}_i^{*}$.) For $i \neq j$, this is a **heteroclinic point**.



Homoclinic tangle for $x_{n+1} = y_n$ and $y_{n+1} = (a+by_n^2)y_n - x_n$ with $a = 2.693$, $b = -104.888$. Blue curve is the stable manifold. Red curve is the unstable manifold. HFP at $(0,0)$. The fact that neither red nor blue curve can self intersect requires them to become increasingly tortured.

But since $\hat{T}_\epsilon^s$ is continuous and invertible, its action on a homoclinic (heteroclinic) point will produce a <u>new</u> homoclinic (heteroclinic) point, ad infinitum! For homoclinic intersections, the result is known as a **homoclinic tangle**.

- **Maps in $d=1$**: $x_{n+1} = f(x_n)$; fixed point $x^* = f(x^*)$

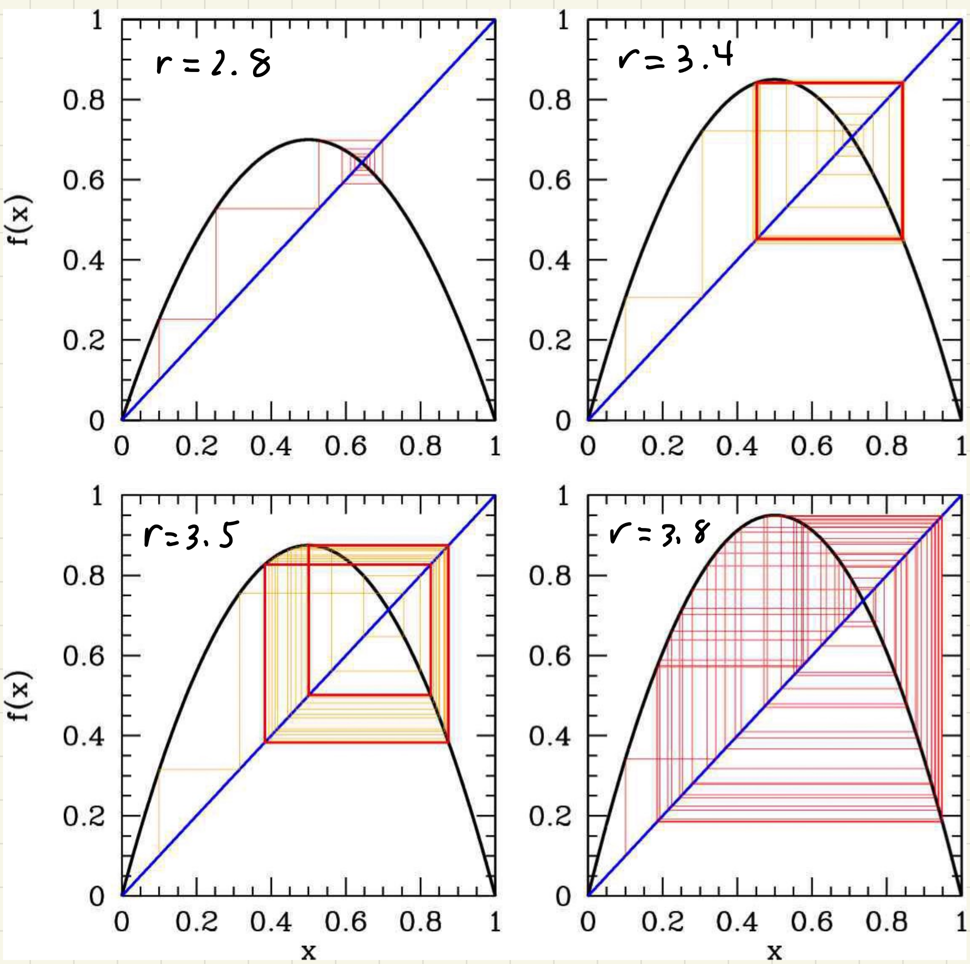
  If $x = x^* + u$, then $u_{n+1} = f'(x^*)\, u_n + \mathcal{O}(u^2)$

  FP is **stable** if $|f'(x^*)| < 1$, **unstable** if $|f'(x^*)| > 1$.

The most studied one-dimensional map is the **logistic map**,

$$f(x) = rx(1-x)$$

on the interval $x \in [0,1]$. Setting $f(x) = x$ we obtain fixed points at $x^* = 0$ and $x^* = 1 - r^{-1}$, where the latter requires $r > 1$. Note $f'(0) = r$, so if $r < 1$ then $x^* = 0$ is stable. If $r > 1$, $x^* = 0$ is unstable, but what about $x^* = 1 - r^{-1}$? Well we have $f'(1-r^{-1}) = 2 - r$, so we conclude $x^* = 1 - r^{-1}$ exists and is stable provided $r \in (1,3)$. What happens for $r > 3$? We can explore further with the help of the **cobweb diagram** below. Sketch $y = x$ and $y = f(x)$. Given x, move vertically to $y = f(x)$, then horizontally to $y = x$, etc.



Cobweb diagram for $f(x) = rx(1-x)$

For $r = 3.4$, $x^* = 1 - r^{-1}$ is unstable, but there is a stable **two-cycle** (x_1, x_2), where

$$x_2 = r x_1 (1 - x_1) \quad , \quad x_1 = r x_2 (1 - x_2)$$

The second iterate of $f(x)$ is

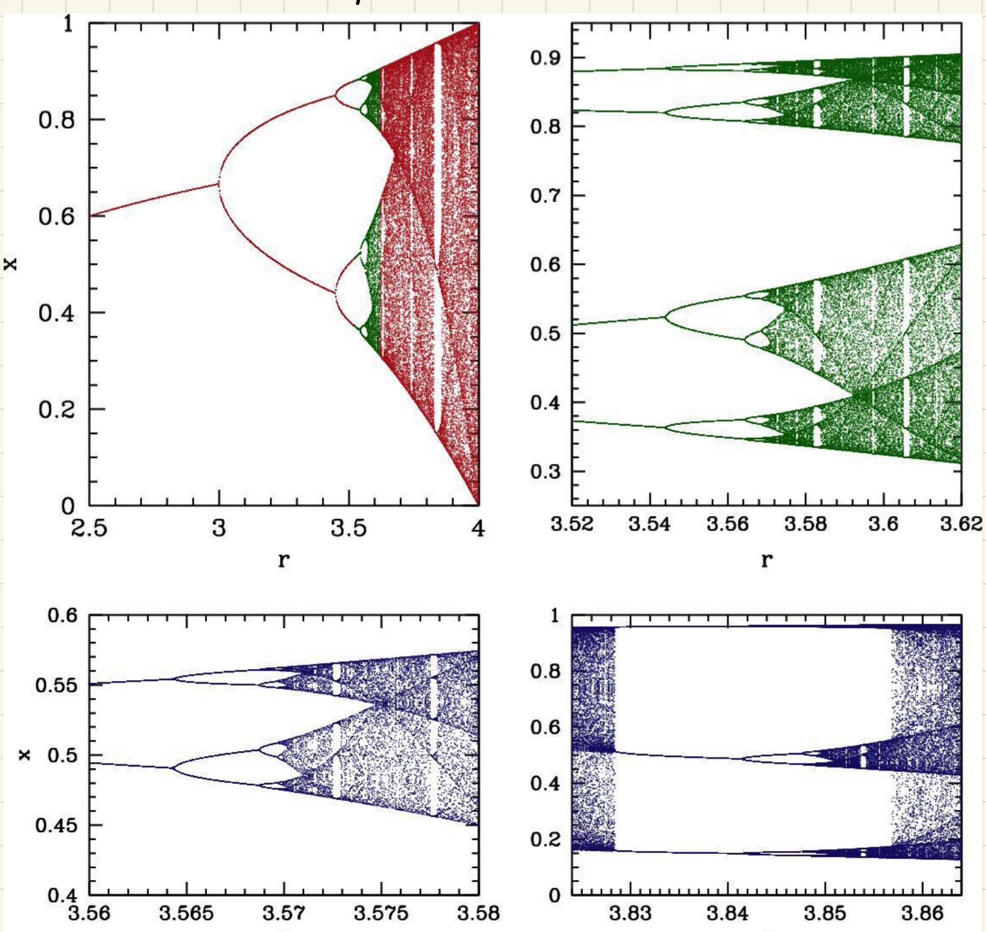
$$f^{(2)}(x) = f(f(x)) = r^2 x (1-x)(1 - rx + rx^2)$$

Setting $f^{(2)}(x) = x$ yields a cubic equation, but $(x - x^*)$ is a factor which we can divide out, yielding

$$x_{1,2} = \frac{1}{2r}\left(1 + r \pm \sqrt{(r+1)(r-3)}\right)$$

How stable is the 2-cycle? We find

$$\left.\frac{d}{dx} f^{(2)}(x)\right|_{x_{1,2}} = -r^2 + 2r + 4$$



Fixed points and cycles for $f(x) = rx(1-x)$

Thus stability of the 2-cycle requires

$$-1 < r^2 - 2r - 4 < 1 \Rightarrow r \in [3, 1+\sqrt{6}]$$

At $r = 1+\sqrt{6} = 3.449\ldots$ there is a bifurcation to a stable 4-cycle (see figure above). The 4-cycle becomes unstable at $r = 3.544\ldots$ and bifurcates into an 8-cycle. This sequence of bifurcations continues:

$$r_1 = 3 \quad , \quad r_2 = 3.4494897\ldots \quad , \quad r_3 = 3.544096\ldots$$

$$r_4 = 3.564407\ldots \quad , \quad r_5 = 3.568759\ldots \quad , \quad r_6 = 3.569692\ldots$$

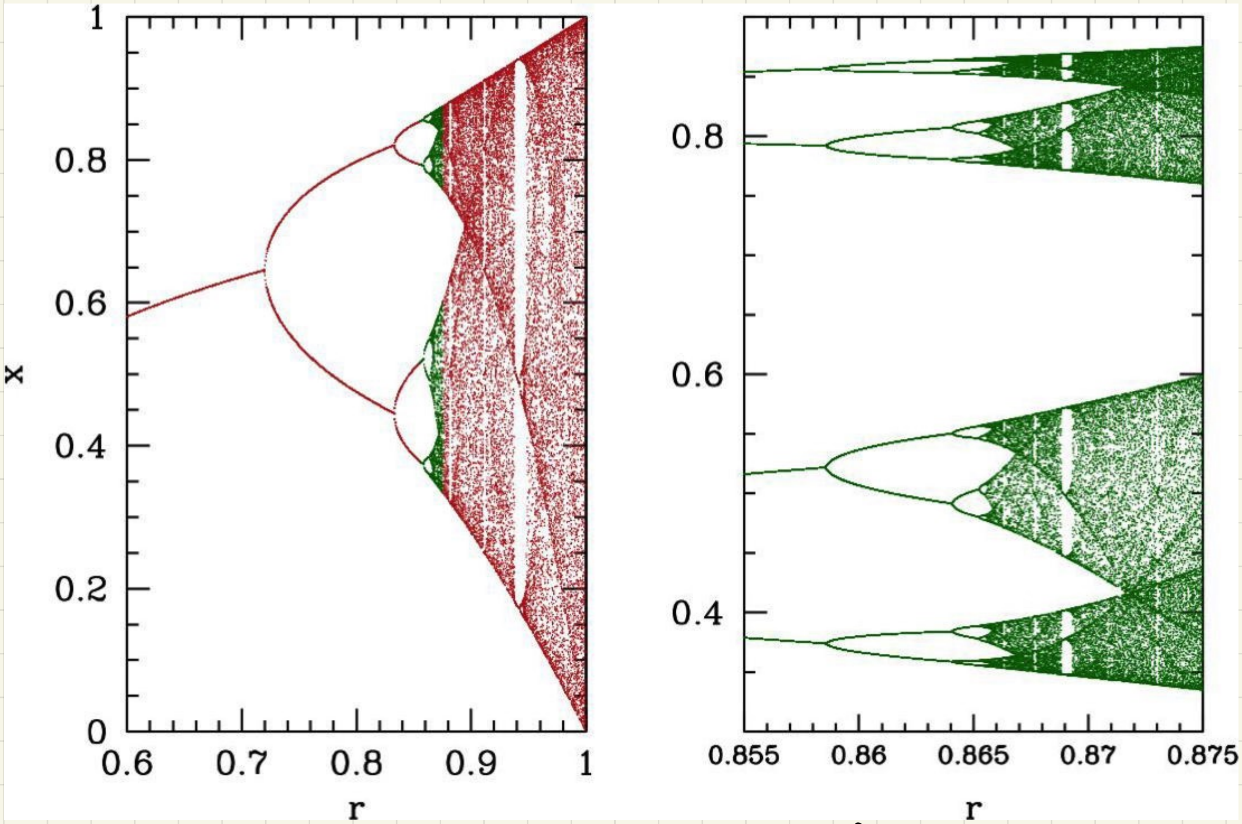
$$r_7 = 3.569891\ldots \quad , \quad r_8 = 3.569934\ldots \quad , \quad \ldots$$

Here r_k is the location of the k^{th} bifurcation from a k-cycle to a (2^k)-cycle. Mitchell Feigenbaum noticed that the sequence $\{r_1, r_2, \ldots\}$ seemed to converge exponentially. Writing

$$r_\infty - r_k \sim \frac{c}{\delta^k} \quad , \quad \delta = \lim_{k \to \infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k}$$

Feigenbaum found

$$r_\infty = 3.5699456\ldots \quad , \quad \delta = 4.669202\ldots \quad , \quad c = 2.637\ldots$$



Iterates of the sine map $f(x) = r\sin(\pi x)$

At $r = r_\infty$, the period doubling cascade diverges in frequency and we enter a regime of **chaos**. A nice way of looking at this is to consider the map for the value $r = 4$. Then defining $x_n = \sin^2\theta_n$ we have

$$x_{n+1} = \sin^2\theta_{n+1} = 4x_n(1-x_n)$$
$$= 4\sin^2\theta_n\cos^2\theta_n = \sin^2(2\theta_n)$$

which is to say $\theta_{n+1} = 2\theta_n$. Now consider the binary decimal expansion of θ_n/π. We start with

$$\frac{\theta_0}{\pi} \equiv \sum_{k=1}^{\infty} \frac{b_k}{2^k} \in [0,1]$$

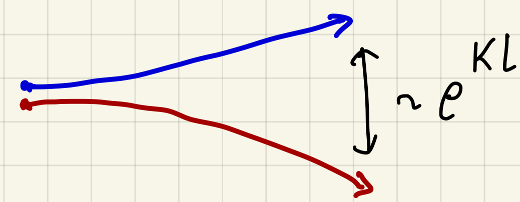
with $b_n \in \{0,1\}$. Thus $\frac{\theta_0}{\pi} = 0.b_1 b_2 b_3 \ldots$ in "binary decimal" form. Under the logistic map, we have $\theta_n = 2^n \theta_0$, and therefore

$$\theta_n = \pi \sum_{n=1}^{\infty} \frac{b_{n+k}}{2^k}$$

Note that we may strip off any integer multiples of π from θ_n since $x_n = \sin^2\theta_n$. Thus,

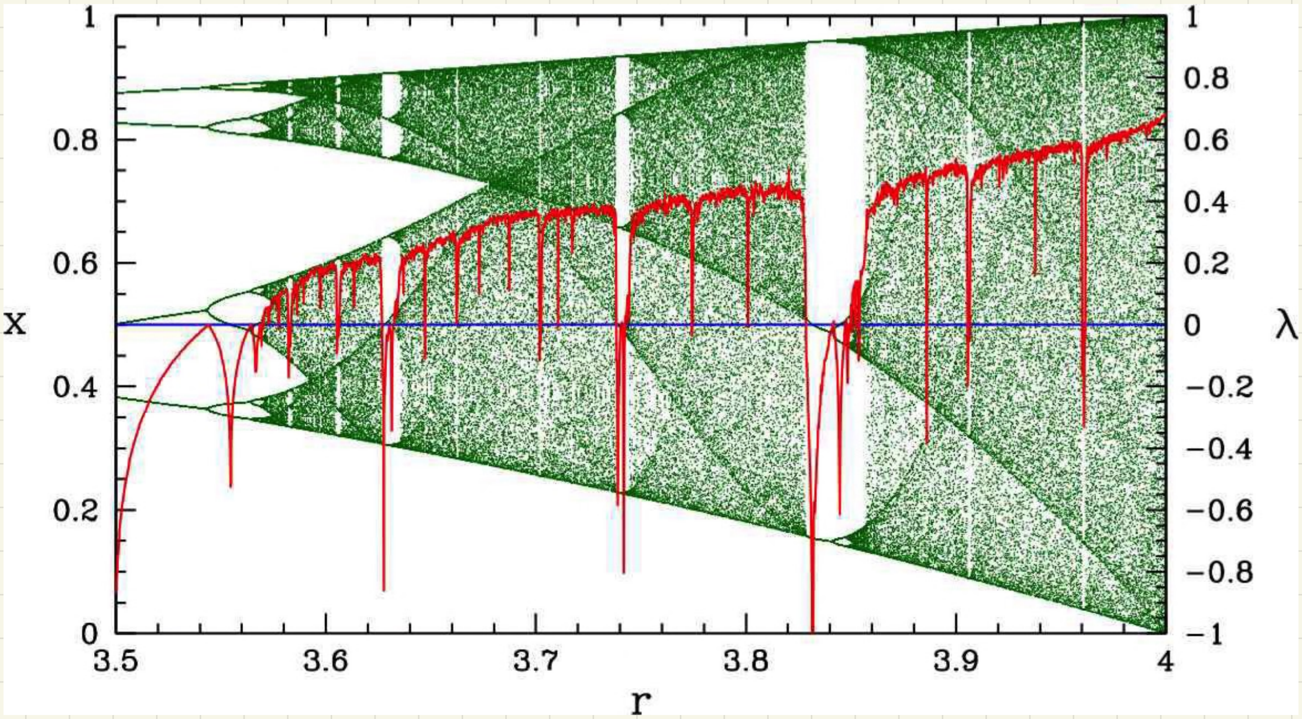
$$\frac{\theta_n}{\pi} = 0.b_{n+1} b_{n+2} b_{n+3} \ldots$$

The logistic map at $r=4$ effectively **shifts the digits** in the binary expansion to the left by one space with each iteration. (The leftmost digit falls off the edge of the world.) Thus, two initial binary expansions of θ_0/π which differ by 2^{-M} will after M iterations differ by $\mathcal{O}(1)$.



Lyapunov exponents

The Lyapunov exponent $\lambda(x)$ for the iterated map $x_{n+1} = f(x_n)$ is defined as



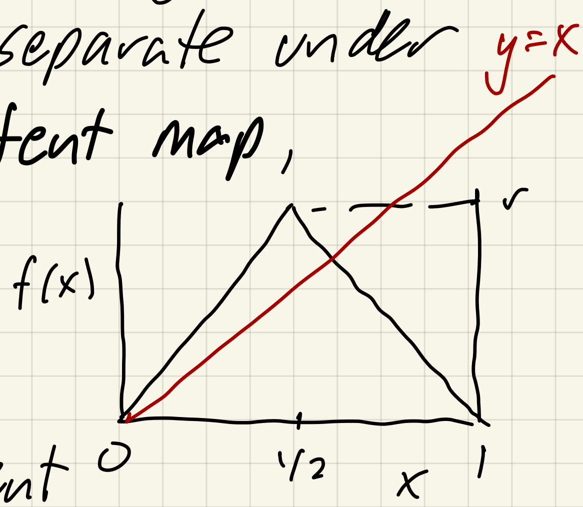
Lyapunov exponent (red) for the logistic map

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log\left| \frac{d f^{(n)}(x)}{dx} \right|$$

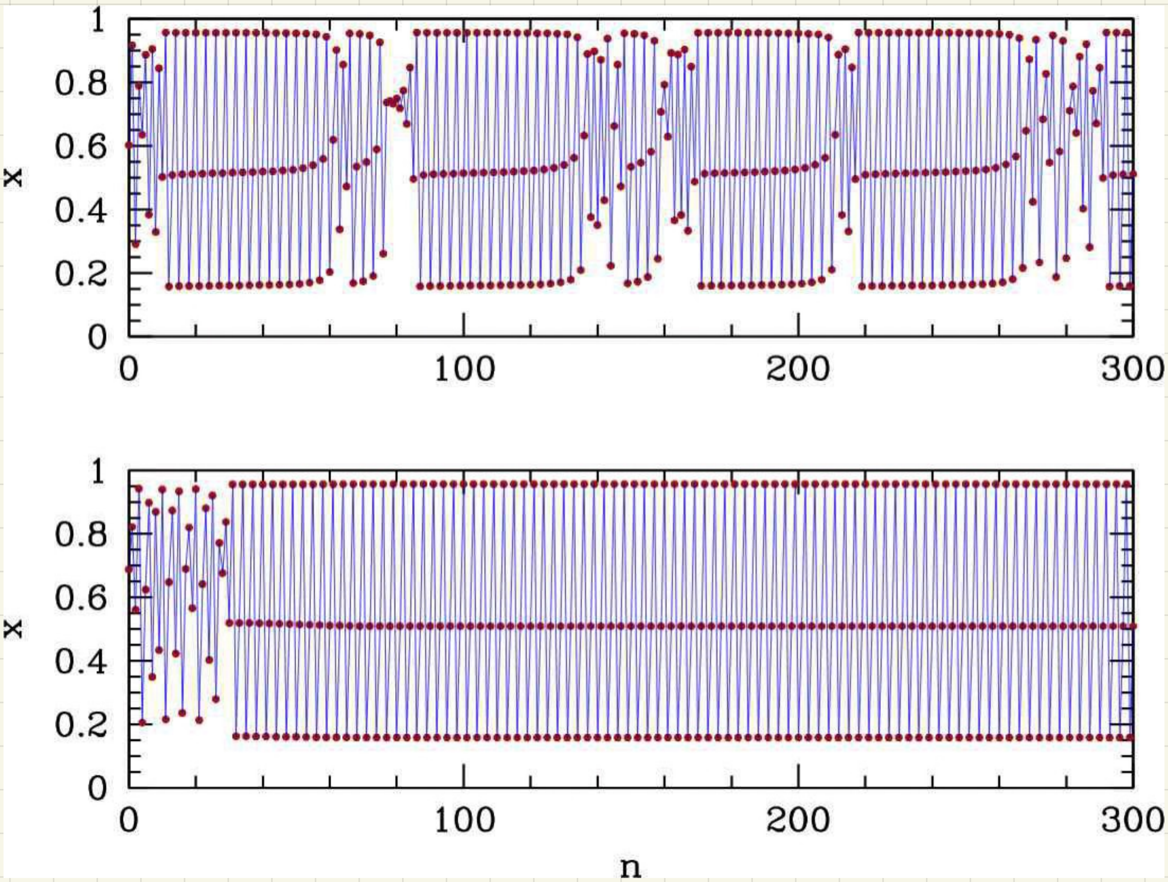
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log |f'(x_j)|$$

If $\mathrm{Re}\,\lambda(x) > 0$, then two nearby initial conditions will exponentially separate under the iterated map. For the **tent map**,

$$f(x) = \begin{cases} 2rx & \text{if } x < \frac{1}{2} \\ 2r(1-x) & \text{if } x \geq \frac{1}{2} \end{cases}$$



one finds $\lambda(x) = \log(2r)$ independent of the position x. Thus $r > \frac{1}{2} \Rightarrow \lambda > 0$.



$r = 3.828$

$r = 3.829$

Intermittency in the logistic map $x_{n+1} = r x_n(1-x_n)$ in the vicinity of the stable 3-cycle, with $r = 3.828$ (top) and $r = 3.829$ (bottom).

Intermittency

Period doubling is not the only route to chaos. Consider the logistic map $x_{n+1} = r x_n(1-x_n)$ for $r = 3.829$, shown in the bottom panel above. There is a stable 3-cycle. But if we reduce the control parameter to $r = 3.828$, the 3-cycle becomes unstable. The map produces an almost stable 3-cycle irregularly interrupted by **bursts**. The average time between bursts scales as a power law: $T(r) \propto (r_c - r)^{-s}$, where s is a critical exponent. Depending on how the Lyapunov exponent $\nu(r)$ behaves in the vicinity of r_c, with $\mathrm{Re}\,\nu(r) > 0$ in the chaotic (bursting) phase, the intermittent behavior is classified as one of three types:

- type I: $\mathrm{Re}\,\nu(r_c) = 0$, $\mathrm{Im}\,\nu(r_c) = 0$
- type II: $\mathrm{Re}\,\nu(r_c) = 0$, $\mathrm{Im}\,\nu(r_c) \neq 0, \pi$ $(n \geq 2)$
- type III: $\mathrm{Re}\,\nu(r_c) = 0$, $\mathrm{Im}\,\nu(r_c) = \pi$

Dynamical Systems (221A S22 course on NLD)

$$\vec{\varphi} = \{\varphi_1, \dots, \varphi_n\} \in M$$

DS : $\dfrac{d\vec{\varphi}}{dt} = \vec{V}(\vec{\varphi})$; $\vec{V}(\vec{\varphi}) \in TM_{\vec{\varphi}}$

$\vec{V} \in TM$

$$\dot{\varphi}_1 = V_1(\varphi_1, \dots \varphi_n)$$
$$\dot{\varphi}_2 = V_2(\varphi_1, \dots, \varphi_n)$$
$$\vdots$$
$$\dot{\varphi}_n = V_n(\varphi_1, \dots, \varphi_n)$$

Fixed points : $\vec{V}(\vec{\varphi}^*) = 0$

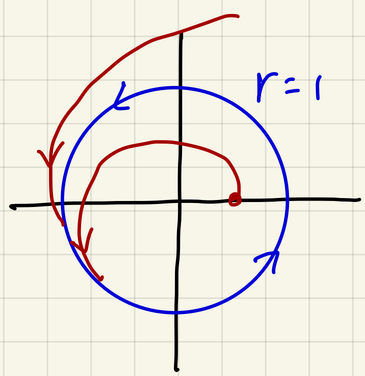
Linearized dynamics in vicinity of $\vec{\varphi}^*$: $\vec{\varphi} = \vec{\varphi}^* + \vec{\epsilon}$

$$\frac{d}{dt}\epsilon_j = \left.\frac{\partial V_j}{\partial \varphi_k}\right|_{\vec{\varphi}^*} \cdot \epsilon_k = R_{jk}(\vec{\varphi}^*)\,\epsilon_k$$

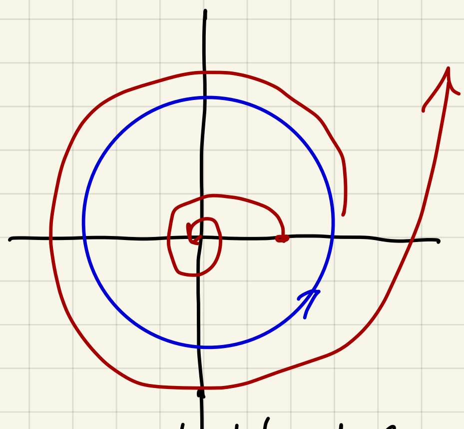
Limit cycles : 2D plane, polar coords (r, θ)

$\dot{r} = a(1-r)$
$\dot{\theta} = 1$

$\dot{r} = a(r-1)$
$\dot{\theta} = 1$



stable LC

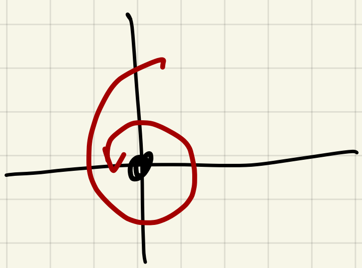


unstable LC

$\dot{r} = a(1-r)^2$
$\dot{\theta} = 1$

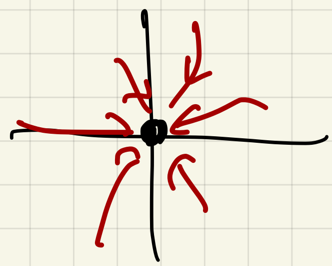
half-stable

Attractors of DSs : SFPs , SLCs .



SPIRAL

$\mathrm{Re}\,\nu < 0$
$\mathrm{Im}\,\nu \neq 0$



NODE

$\mathrm{Re}\,\nu < 0$
$\mathrm{Im}\,\nu = 0$