PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #6

(1) Consider a spin-2 Ising model with Hamiltonian

$$\hat{H} = -\tfrac{1}{2} \sum_{i,j} J_{ij}\, S_i\, S_j - H \sum_i S_i$$

where $S_i \in \{-2, -1, 0, 1, 2\}$. The system is on a simple cubic lattice, with nearest neighbor coupling $J_1/k_{\mathrm{B}} = 40\,\mathrm{K}$ and next-nearest neighbor coupling $J_2/k_{\mathrm{B}} = 10\,\mathrm{K}$.

(a) Find the mean field free energy per site $f(\theta, h, m)$, where $\theta = k_{\mathrm{B}}T/\hat{J}(0)$, $h = H/\hat{J}(0)$, $m = \langle S_i \rangle$, and $f = F/N\hat{J}(0)$.

(b) Find the mean field equation for m.

(c) Setting $h = 0$, find θ_{c}. What is T_{c}?

(d) Find the linear magnetic susceptibility $\chi(\theta)$ for $\theta > \theta_{\mathrm{c}}$.

(e) For $0 < \theta_{\mathrm{c}} - \theta \ll 1$ and $h = 0$, the magnetization is of the form $m = A(\theta_{\mathrm{c}} - \theta)^{1/2}$. Find the coefficient A.

(2) Consider an Ising model on a square lattice with Hamiltonian

$$\hat{H} = -J \sum_{i \in \mathrm{A}} {\sum_{j \in \mathrm{B}}}' S_i\, \sigma_j\ ,$$

where the sum is over all nearest-neighbor pairs, such that i is on the A sublattice and j is on the B sublattice (this is the meaning of the prime on the j sum), as depicted in Fig. 1. The A sublattice spins take values $S_i \in \{-1, 0, +1\}$, while the B sublattice spins take values $\sigma_j \in \{-1, +1\}$.

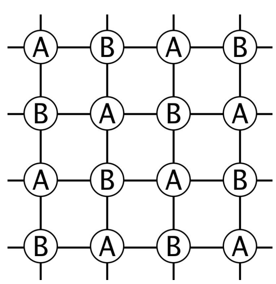

Figure 1: The square lattice and its A and B sublattices.

(a) Make the mean field assumptions $\langle S_i \rangle = m_{\mathrm{A}}$ for $i \in \mathrm{A}$ and $\langle \sigma_j \rangle = m_{\mathrm{B}}$ for $j \in \mathrm{B}$. Find the mean field free energy $F(T, N, m_{\mathrm{A}}, m_{\mathrm{B}})$. Adimensionalize as usual, writing $\theta \equiv k_{\mathrm{B}}T/zJ$ (with $z = 4$ for the square lattice) and $f = F/zJN$. Then write $f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}})$.

(b) Write down the two mean field equations (one for m_{A} and one for m_{B}).

(c) Expand the free energy $f(\theta, m_{\rm A}, m_{\rm B})$ up to fourth order in the order parameters $m_{\rm A}$ and $m_{\rm B}$. You may find the following useful:

$$\ln\big(2\cosh x\big) = \ln 2 + \frac{x^2}{2} - \frac{x^4}{12} + \mathcal{O}(x^6) \quad , \quad \ln\big(1 + 2\cosh x\big) = \ln 3 + \frac{x^2}{3} - \frac{x^4}{36} + \mathcal{O}(x^6) \; .$$

(d) Show that the part of $f(\theta, m_{\rm A}, m_{\rm B})$ which is quadratic in $m_{\rm A}$ and $m_{\rm B}$ may be written as a quadratic form, *i.e.*

$$f(\theta, m_{\rm A}, m_{\rm B}) = f_0 + \tfrac{1}{2}\begin{pmatrix} m_{\rm A} & m_{\rm B} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} m_{\rm A} \\ m_{\rm B} \end{pmatrix} + \mathcal{O}\big(m_{\rm A}^4, m_{\rm B}^4\big) \; ,$$

where the matrix M is symmetric, with components $M_{aa'}$ which depend on θ. The critical temperature $\theta_{\rm c}$ is identified as the largest value of θ for which $\det M(\theta) = 0$. Find $\theta_{\rm c}$ and explain why this is the correct protocol to determine it.

(3) Consider the ferromagnetic XY model, with

$$\hat{H} = -\sum_{i<j} J_{ij} \, \cos(\phi_i - \phi_j) - H \sum_i \cos\phi_i \; .$$

Defining $z_i \equiv \exp(i\phi_i)$, write $z_i = \langle z_i \rangle + \delta z_i$ with

$$\langle z_i \rangle = m \, e^{i\alpha} \; .$$

(a) Assuming $H > 0$, what should you take for α?

(b) Making this choice for α, find the mean field free energy using the 'neglect of fluctuations' method. *Hint*: Note that $\cos(\phi_i - \phi_j) = \mathsf{Re}\,(z_i z_j^*)$.

(c) Find the self-consistency equation for m.

(d) Find $T_{\rm c}$.

(e) Find the mean field critical behavior for $m(T, H = 0)$, $m(T = T_{\rm c}, H)$, $C_V(T, H = 0)$, and $\chi(T, H = 0)$, and identify the critical exponents α, β, γ, and δ.

(4) Consider the free energy

$$f(\theta, m) = f_0 + \tfrac{1}{2} a m^2 + \tfrac{1}{4} b m^4 + \tfrac{1}{8} d m^8$$

with $d > 0$. Note there is an octic term but no sextic term. Derive results corresponding to those in fig. 7.17 of the lecture notes. Find the equation of the first order line in the $(a/d, b/d)$ plane. Also identify the region in parameter space where there exist metastable local minima in the free energy (curve E in fig. 7.17).