PHYSICS 140B : STATISTICAL PHYSICS
MIDTERM EXAMINATION

(1) Consider the single particle density of states

$$g(\varepsilon) = A\,\varepsilon\,\Theta(\varepsilon) + B\,\Theta(\varepsilon - \Delta) = \begin{cases} A\,\varepsilon & \text{if } 0 \le \varepsilon < \Delta \\ A\,\varepsilon + B & \text{if } \varepsilon \ge \Delta \end{cases} \quad .$$

Here, A, B, and Δ are positive constants. The following integrals may be useful:

$$\int\limits_0^\infty \! d\varepsilon \frac{\varepsilon^\alpha}{z^{-1}\,e^{\varepsilon/k_B T} - 1} = (k_B T)^{1+\alpha}\,\mathrm{Li}_{1+\alpha}(z) \quad , \quad \int\limits_0^\infty \! d\varepsilon\,\varepsilon^\alpha \ln\!\big[1 - z\,e^{-\varepsilon/k_B T}\big] = -(k_B T)^{1+\alpha}\,\mathrm{Li}_{2+\alpha}(z)$$

where $\mathrm{Li}_s(z) = \sum_{m=1}^\infty z^m/m^s$. Note $\mathrm{Li}_1(z) = -\ln(1-z)$.

(a) Assuming the particles have photon statistics, find an expression for the number density $n(T)$. `[10 points]`

The Bose occupancy function is $\nu(\varepsilon) = 1/\big(e^{(\varepsilon-\mu)/k_B T} - 1\big)$. With $\mu = 0$ for photons, we have

$$\begin{aligned} n(T) = \int\limits_0^\infty \! d\varepsilon\,\frac{g(\varepsilon)}{e^{\varepsilon/k_B T} - 1} &= A\!\int\limits_0^\infty \! d\varepsilon\,\frac{\varepsilon}{e^{\varepsilon/k_B T} - 1} + B\!\int\limits_0^\infty \! d\varepsilon\,\frac{1}{e^{(\varepsilon+\Delta)/k_B T} - 1} \\ &= A\,(k_B T)^2\,\mathrm{Li}_2(1) + B\,k_B T\,\mathrm{Li}_1\big(e^{-\Delta/k_B T}\big) \\ &= \frac{\pi^2}{6}\,A\,(k_B T)^2 - B\,k_B T\,\ln\!\big(1 - e^{-\Delta/k_B T}\big) \quad , \end{aligned}$$

using $\mathrm{Li}_s(1) = \zeta(s)$ and $\zeta(2) = \pi^2/6$.

(b) Assuming the particles are bosons, find an equation which relates the critical temperature T_c for Bose condensation to the number density n. `[10 points]`

At $T = T_c$ in a BEC, the chemical potential (in the thermodynamic limit) is pinned infinitesimally below the lowest single particle energy ε_0, which in our case is $\mu = \varepsilon_0 = 0$. The condensate fraction at T_c is $n_0 = 0$. Thus we have

$$n(T_c) = n = \int\limits_0^\infty \! d\varepsilon\,\frac{g(\varepsilon)}{e^{\varepsilon/k_B T_c} - 1} = \frac{\pi^2}{6}\,A\,(k_B T_c)^2 - B\,k_B T_c\,\ln\!\big(1 - e^{-\Delta/k_B T_c}\big) \quad .$$

It is easy to see that $n(T_c)$ is a monotonically increasing function of T_c, and hence $T_c(n)$ is a monotonically increasing function of the density n.

(c) For $T > T_c$, find a closed form expression for $n(T, z)$, where $z = \exp(\mu/k_B T)$ is the fugacity. `[10 points]`

We have

$$n(T,z) = \int_0^\infty \! d\varepsilon \, \frac{g(\varepsilon)}{z^{-1}\, e^{\varepsilon/k_B T} - 1} = A \int_0^\infty \! d\varepsilon \, \frac{\varepsilon}{z^{-1}\, e^{\varepsilon/k_B T} - 1} + B \int_0^\infty \! d\varepsilon \, \frac{1}{z^{-1}\, e^{(\varepsilon+\Delta)/k_B T} - 1}$$

$$= A\,(k_B T)^2\, \mathsf{Li}_2(z) + B\, k_B T\, \mathsf{Li}_1\big(z\, e^{-\Delta/k_B T}\big)$$

$$= A\,(k_B T)^2\, \mathsf{Li}_2(z) - B\, k_B T\, \ln\big(1 - z\, e^{-\Delta/k_B T}\big) \quad ,$$

(d) For $T < T_c$, find an expression for $n(T, n_0)$, where n_0 is the condensate number density. `[10 points]`

We set $z = 1$ ($\mu = 0$) and include a contribution from the condensate density, thus

$$n(T, n_0) = n_0 + \int_0^\infty \! d\varepsilon \, \frac{g(\varepsilon)}{e^{\varepsilon/k_B T} - 1}$$

$$= n_0 + \frac{\pi^2}{6}\, A\,(k_B T)^2 - B\, k_B T\, \ln\big(1 - e^{-\Delta/k_B T}\big) \quad .$$

(e) For $T < T_c$, find $p(T)$. `[10 points]`

We have

$$p(T) = -k_B T \int_0^\infty \! d\varepsilon \, g(\varepsilon)\, \ln\big[1 - e^{-\varepsilon/k_B T}\big]$$

$$= -A\, k_B T \int_0^\infty \! d\varepsilon \, \varepsilon\, \ln\big[1 - e^{-\varepsilon/k_B T}\big] - B\, k_B T \int_0^\infty \! d\varepsilon \, \ln\big[1 - e^{-(\varepsilon+\Delta)/k_B T}\big]$$

$$= A\, \zeta(3)\, (k_B T)^3 + B\, (k_B T)^2\, \mathsf{Li}_2\big(e^{-\Delta/k_B T}\big) \quad .$$

(f) For $T > T_c$, find an expression for $p(T, n)$. `[50 quatloos extra credit]`

We have

$$p(T,z) = -k_B T \int_0^\infty \! d\varepsilon \, g(\varepsilon)\, \ln\big[1 - z\, e^{-\varepsilon/k_B T}\big]$$

$$= -A\, k_B T \int_0^\infty \! d\varepsilon \, \varepsilon\, \ln\big[1 - z\, e^{-\varepsilon/k_B T}\big] - B\, k_B T \int_0^\infty \! d\varepsilon \, \ln\big[1 - z\, e^{-(\varepsilon+\Delta)/k_B T}\big]$$

$$= A\, (k_B T)^3\, \mathsf{Li}_3(z) + B\, (k_B T)^2\, \mathsf{Li}_2\big(z\, e^{-\Delta/k_B T}\big) \quad .$$

(2) Consider $S = \frac{1}{2}$ fermions with the relativistic dispersion

$$\varepsilon(\boldsymbol{p}) = \sqrt{c^2 p^2 + m^2 c^4} \quad ,$$

in $d = 2$ space dimensions, where $\boldsymbol{p} = \hbar \boldsymbol{k}$ is the momentum.

(a) Find the density of states $g(\varepsilon)$. Don't forget to include the appropriate step function to indicate the energy below which $g(\varepsilon)$ vanishes. `[10 points]`

The density of states is obtained from the relation

$$g(\varepsilon)\, d\varepsilon = \mathsf{g}\, \frac{d^2 p}{h^2} = \frac{1}{\pi \hbar^2}\, p\, dp \quad \Rightarrow \quad g(\varepsilon) = \frac{1}{\pi \hbar^2}\, p\, \frac{dp}{d\varepsilon} = \frac{1}{2\pi \hbar^2}\, \frac{d(p^2)}{d\varepsilon} \quad ,$$

and from

$$p^2 = \frac{1}{c^2}\left(\varepsilon^2 - m^2 c^2\right) \quad ,$$

we obtain

$$g(\varepsilon) = \frac{\varepsilon}{\pi \hbar^2 c^2}\, \Theta(\varepsilon - mc^2) \quad .$$

(b) Find the Fermi momentum $p_{\rm F}(n)$. `[10 points]`

The number density is

$$n = \mathsf{g} \int \frac{d^2 p}{h^2}\, \Theta(p_{\rm F} - p) = \frac{2}{h^2} \cdot \pi p_{\rm F}^2 = \frac{p_{\rm F}^2}{2\pi \hbar^2} \quad ,$$

and thus

$$p_{\rm F}(n) = \hbar \sqrt{2\pi n} \quad .$$

(c) Find the second virial coefficient $B_2(T)$. `[15 points]`

The following integral may be useful: $\int\limits_a^\infty dx\, x\, e^{-x} = (1 + a)\, e^{-a}$.

Recall that $B_2 = -C_2 / 2C_1^2$, where

$$C_j(T) = (-1)^{j-1} \int\limits_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon)\, e^{-j\varepsilon / k_{\rm B} T} = \frac{(-1)^{j-1}}{\pi \hbar^2 c^2} \int\limits_{mc^2}^{\infty} d\varepsilon\, \varepsilon\, e^{-j\varepsilon / k_{\rm B} T}$$
$$= \frac{(-1)^{j-1}}{\pi \hbar^2 c^2} \left(\frac{k_{\rm B} T}{j}\right)^2 \left(1 + \frac{jmc^2}{k_{\rm B} T}\right) e^{-jmc^2 / k_{\rm B} T} \quad .$$

Thus,

$$B_2(T) = -\frac{C_2(T)}{2\, C_1^2(T)} = \frac{\pi \hbar^2 c^2}{8\, (k_{\rm B} T)^2} \cdot \frac{k_{\rm B} T + 2mc^2}{(k_{\rm B} T + mc^2)^2} \quad .$$

(d) Find the chemical potential $\mu(T,n)$, valid to order T^2. [15 points]

We have

$$\mu(T=0,n) = \varepsilon(p_{\rm F}) = \sqrt{2\pi\hbar^2 c^2 n + m^2 c^4}$$

and

$$\delta\mu(T,n) = -\frac{\pi^2}{6}\,(k_{\rm B}T)^2\,\frac{g'(\varepsilon_{\rm F})}{g(\varepsilon_{\rm F})} = -\frac{\pi^2}{6}\,\frac{(k_{\rm B}T)^2}{\varepsilon_{\rm F}} = -\frac{\pi^2}{6}\,\frac{(k_{\rm B}T)^2}{\sqrt{2\pi\hbar^2 c^2 n + m^2 c^4}} \quad .$$

Thus,

$$\mu(T) = \sqrt{2\pi\hbar^2 c^2 n + m^2 c^4}\,\left\{1 - \frac{\pi^2}{6}\,\frac{(k_{\rm B}T)^2}{2\pi\hbar^2 c^2 n + m^2 c^4} + \ldots\right\} \quad .$$

(e) Find $n(T,z)$. [50 quatloos extra credit]

For fermions with our DOS, we have

$$\begin{aligned} n(T,z) &= \int\limits_0^\infty \! d\varepsilon\,\frac{g(\varepsilon)}{z^{-1}\,e^{\varepsilon/k_{\rm B}T}+1} \\ &= \frac{1}{\pi\hbar^2 c^2}\int\limits_{mc^2}^\infty \! d\varepsilon\;\varepsilon\sum_{j=1}^\infty z^j\,e^{-j\varepsilon/k_{\rm B}T} = \frac{1}{\pi}\left(\frac{k_{\rm B}T}{\hbar c}\right)^{\!2}\sum_{j=1}^\infty \frac{z^j}{j^2}\left(1+\frac{jmc^2}{k_{\rm B}T}\right)e^{-jmc^2/k_{\rm B}T} \\ &= \frac{1}{\pi}\left(\frac{k_{\rm B}T}{\hbar c}\right)^{\!2}\left\{\mathrm{Li}_2\big(z\,e^{-mc^2/k_{\rm B}T}\big) - \frac{mc^2}{k_{\rm B}T}\,\ln\!\big[1 - z\,e^{-mc^2/k_{\rm B}T}\big]\right\} \quad . \end{aligned}$$