PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #3 SOLUTIONS

(1) Consider a two-dimensional gas of fermions which obey the dispersion relation

$$\varepsilon(\boldsymbol{k}) = \varepsilon_0 \Big((k_x^2 + k_y^2)\, a^2 + \tfrac{1}{2}(k_x^4 + k_y^4)\, a^4 \Big) \quad .$$

Sketch, on the same plot, the Fermi surfaces for $\varepsilon_{\rm F} = 0.1\,\varepsilon_0$, $\varepsilon_{\rm F} = \varepsilon_0$, and $\varepsilon_{\rm F} = 10\,\varepsilon_0$.

It is convenient to adimensionalize, writing

$$x \equiv k_x a \quad , \quad y \equiv k_y a \quad , \quad \nu \equiv \frac{\varepsilon}{\varepsilon_0} \quad .$$

Then the equation for the Fermi surface becomes

$$x^2 + y^2 + \tfrac{1}{2}x^4 + \tfrac{1}{2}y^4 = \nu \quad .$$

In other words, we are interested in the *level sets* of the function $\nu(x,y) \equiv x^2 + y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4$. When ν is small, we can ignore the quartic terms, and we have an isotropic dispersion, with $\nu = x^2 + y^2$. *I.e.* we can write $x = \nu^{1/2}\cos\theta$ and $y = \nu^{1/2}\sin\theta$. The quartic terms give a contribution of order ν^4, which is vanishingly small compared with the quadratic term in the $\nu \to 0$ limit. When $\nu \sim \mathcal{O}(1)$, the quadratic and quartic terms in the dispersion are of the same order of magnitude, and the continuous $\mathsf{O}(2)$ symmetry, namely the symmetry under rotation by any angle, is replaced by a discrete symmetry group, which is the group of the square, known as C_{4v} in group theory parlance. This group has eight elements:

$$\big\{ \mathbb{I}\,,\ R\,,\ R^2\,,\ R^3\,,\ \sigma\,,\ \sigma R\,,\ \sigma R^2\,,\ \sigma R^3 \big\}$$

Here R is the operation of counterclockwise rotation by $90°$, sending $(x\,,\,y)$ to $(-y\,,\,x)$, and σ is reflection in the y-axis, which sends $(x\,,\,y)$ to $(-x\,,\,y)$. One can check that the function $\nu(x,y)$ is invariant under any of these eight operations from C_{4v}.

Explicitly, we can set $y = 0$ and solve the resulting quadratic equation in x^2 to obtain the maximum value of x, which we call $a(\nu)$. One finds

$$\tfrac{1}{2}x^4 + x^2 - \nu = 0 \qquad \Longrightarrow \qquad a = \sqrt{\sqrt{1+2\nu} - 1} \quad .$$

So long as $x \in \{-a, a\}$, we can solve for $y(x)$:

$$y(x) = \pm\sqrt{\sqrt{1 + 2\nu - 2x^2 - x^4} - 1} \quad .$$

A sketch of the level sets, showing the evolution from an isotropic (*i.e.* circular) Fermi surface at small ν, to surfaces with discrete symmetries, is shown in fig. 1.

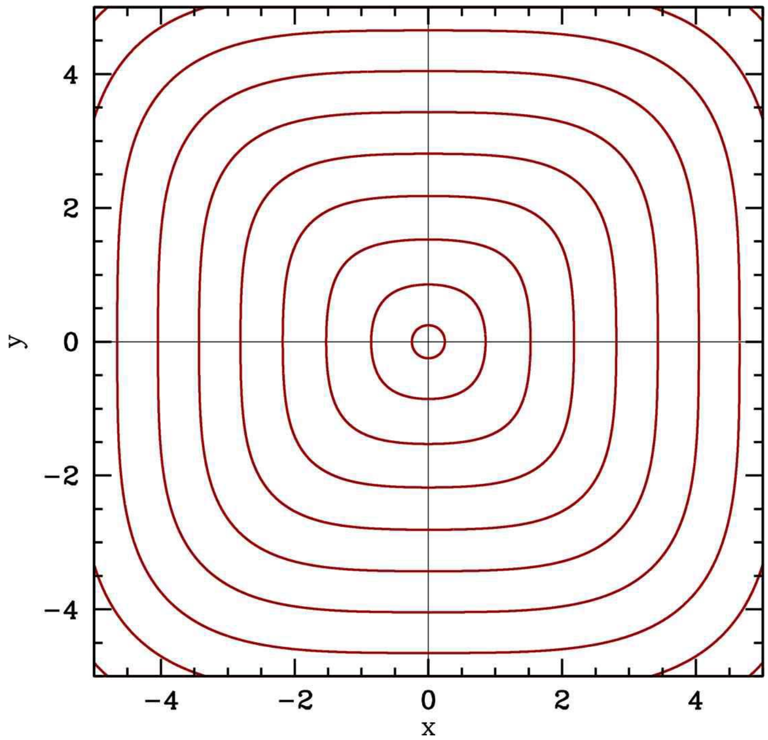

Figure 1: Level sets of the function $\nu(x,y) = x^2 + y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4$ for $\nu = (\frac{1}{2}n)^4$, with positive integer n.

(2) Using the Sommerfeld expansion, compute the heat capacity for a two-dimensional electron gas, to lowest nontrivial order in the temperature T.

In the notes, in section 4.7.6, we obtained the result

$$\frac{E}{V} = \int\limits_{-\infty}^{\varepsilon_{\rm F}} d\varepsilon\, g(\varepsilon)\, \varepsilon + \frac{\pi^2}{6}\,(k_{\rm B}T)^2\, g(\varepsilon_{\rm F}) + \mathcal{O}(T^4) \quad .$$

This entails a heat capacity of $C_{V,N} = V \cdot \frac{1}{3}\pi^2 k_{\rm B}\, g(\varepsilon_{\rm F}) \cdot k_{\rm B}T$. The density of states at the Fermi level, $g(\varepsilon_{\rm F})$, is easily found to be

$$g(\varepsilon_{\rm F}) = \frac{d}{2} \cdot \frac{n}{\varepsilon_{\rm F}} \quad .$$

Thus,

$$C_{V,N} = N \cdot \frac{d\,\pi^2}{6}\, k_{\rm B} \cdot \left(\frac{k_{\rm B}T}{\varepsilon_{\rm F}}\right) ,$$

a form which is valid in any spatial dimension d.

(3) ^{3}He atoms consist of an odd number of fermions (two electrons, two protons, and one neutron), and hence is itself a fermion. Consider a kilomole of ^{3}He atoms at standard temperature and pressure ($T = 293,\ \text{K}$, $p = 1\,\text{atm}$).

(a) What is the Fermi temperature of the gas? Assume $z \ll 1$ and justify this in part (b).

Assuming the gas is essentially classical (this will be justified shortly), we find the gas density using the ideal gas law:

$$n = \frac{p}{k_{\text{B}}T} = \frac{1.013 \times 10^5\,\text{Pa}}{(1.38 \times 10^{-23}\,\text{J/K})(293\,\text{K})} = 2.51 \times 10^{25}\,\text{m}^{-3} \quad .$$

It is convenient to compute the rest energy of a ^{3}He atom. The mass is $3.016\,\text{amu}$ (look it up on Google), hence

$$m_3\, c^2 = 3.016 \cdot (931.5\,\text{MeV}) = 2.809\,\text{GeV} \quad .$$

For the conversion of amu to MeV/c^2, again try googling. We'll then need $\hbar c = 1973\,\text{eV} \cdot \text{Å}$. (I remember 1973 because that was the summer I won third prize in an archery contest at Camp Mahakeno.) Thus,

$$\begin{aligned}\varepsilon_{\text{F}} &= \frac{(\hbar c)^2}{2m_3\, c^2} \cdot (3\pi^2 n)^{2/3} = \frac{(1973\,\text{eV} \cdot 10^{-10}\,\text{m})^2}{2.809 \times 10^9\,\text{eV}} \cdot (3\pi^2 \cdot 2.51 \times 10^{25}\,\text{m}^{-3})^{2/3} \\ &= 1.14 \times 10^{-5}\,\text{eV} \quad .\end{aligned}$$

Now with $k_{\text{B}} = 86.2\,\mu\text{eV/K}$, we have $T_{\text{F}} = \varepsilon_{\text{F}}/k_{\text{B}} = 0.13\,\text{K}$.

(b) Calculate $\mu/k_{\text{B}}T$ and $z = \exp(\mu/k_{\text{B}}T)$.

Within the GCE, the fugacity is given by $z = n\lambda_T^3$. The thermal wavelength is

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mk_{\text{B}}T}\right)^{1/2} = \left(\frac{2\pi \cdot (1973\,\text{eV} \cdot \text{Å})^2}{(2.809 \times 10^9\,\text{eV}) \cdot (86.2 \times 10^{-6}\,\text{eV/K}) \cdot (293\,\text{K})}\right)^{1/2} = 0.587\,\text{Å} \ ,$$

hence

$$z = n\lambda_T^3 = (2.51 \times 10^{-5}\,\text{Å}^{-3}) \cdot (0.587\,\text{Å})^3 = 5.08 \times 10^{-6} \quad .$$

Thus,

$$\frac{\mu}{k_{\text{B}}T} = \ln z = -12.2 \qquad , \qquad z = e^{\mu/k_{\text{B}}T} = 5.08 \times 10^{-6} \quad .$$

(c) Find the average occupancy $n(\varepsilon)$ of a single particle state with energy $\frac{3}{2}k_{\text{B}}T$.

To find the occupancy $f(\varepsilon - \mu)$, we note $\varepsilon - \mu = \left[\frac{3}{2} - (-12.2)\right] k_{\text{B}}T = 13.7\,k_{\text{B}}T$, in which case

$$n(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/k_{\text{B}}T} + 1} = \frac{1}{e^{13.7} + 1} = 1.12 \times 10^{-6} \quad .$$

(4) For ideal Fermi gases in $d = 1$, 2, and 3 dimensions, compute at $T = 0$ the average energy per particle E/N in terms of the Fermi energy $\varepsilon_{\rm F}$.

The number of particles is

$$N = \mathsf{g}\, V \!\int\! \frac{d^d\!k}{(2\pi)^d}\, \Theta(k_{\rm F} - k) = V \cdot \frac{\mathsf{g}\,\Omega_d}{(2\pi)^d}\, \frac{k_{\rm F}^d}{d}\ ,$$

where g is the internal degeneracy and Ω_d is the surface area of a sphere in d dimensions. The total energy is

$$E = \mathsf{g}\, V \!\int\! \frac{d^d\!k}{(2\pi)^d}\, \frac{\hbar^2 k^2}{2m}\, \Theta(k_{\rm F} - k) = V \cdot \frac{\mathsf{g}\,\Omega_d}{(2\pi)^d}\, \frac{k_{\rm F}^d}{d+2} \cdot \frac{\hbar^2 k_{\rm F}^2}{2m} \quad .$$

Therefore,

$$\frac{E}{N} = \frac{d}{d+2}\, \varepsilon_{\rm F} \quad .$$

(5) Obtain numerical estimates for the Fermi energy (in eV) and the Fermi temperature (in Kelvins) for the following systems:

(a) conduction electrons in silver, lead, and aluminum

The Fermi energy for ballistic dispersion is given by

$$\varepsilon_{\rm F} = \frac{\hbar^2}{2m^*}\, (3\pi^2 n)^{2/3}\ ,$$

where m^* is the effective mass, which one can assume is the electron mass $m = 9.11 \times 10^{-28}\,$g. The electron density is given by the number of valence electrons of the atom divided by the volume of the unit cell. A typical unit cell volume is on the order of $30\,\text{Å}^3$, and if we assume one valence electron per atom we obtain a Fermi energy of $\varepsilon_{\rm F} = 3.8\,$eV, and hence a Fermi temperature of $3.8\,\text{eV}/(86.2 \times 10^{-6}\,\text{eV/K}) = 4.4 \times 10^4\,$K. This sets the overall scale. For detailed numbers, one can examine table 2.1 in *Solid State Physics* by Ashcroft and Mermin. One finds

$$T_{\rm F}(\text{Ag}) = 6.38 \times 10^4\,\text{K} \quad ; \quad T_{\rm F}(\text{Pb}) = 11.0 \times 10^4\,\text{K} \quad ; \quad T_{\rm F}(\text{Al}) = 13.6 \times 10^4\,\text{K} \quad .$$

(b) nucleons in a heavy nucleus, such as ^{200}Hg

Nuclear densities are of course much higher. In the literature one finds the relation $R \sim A^{1/3}\, r_0$, where R is the nuclear radius, A is the number of nucleons (*i.e.* the atomic mass number), and $r_0 \simeq 1.2\ \text{fm} = 1.2 \times 10^{-15}\,$m Under these conditions, the nuclear density is on the order of $n \sim 3A/4\pi R^3 = 3/4\pi r_0^3 = 1.4 \times 10^{44}\,\text{m}^{-3}$. With the mass of the proton $m_{\rm p} = 938\,\text{MeV}/c^2$ we find $\varepsilon_{\rm F} \sim 30\,$MeV for the nucleus, corresponding to a temperature of roughly $T_{\rm F} \sim 3.5 \times 10^{11}\,$K.