PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #5 SOLUTIONS

(1) DC Comics superhero Clusterman and his naughty dog Henry are shown in fig. 1. Clusterman, as his name connotes, is a connected diagram, but the diagram for Henry contains some disconnected pieces.

(a) Interpreting the diagrams as arising from the Mayer cluster expansion, compute the symmetry factor s_γ for Clusterman.

(b) What is the *total* symmetry factor for Henry and his disconnected pieces? What would the answer be if, unfortunately, another disconnected piece of the same composition were to be found?

(c) What is the lowest order virial coefficient to which Clusterman contributes?

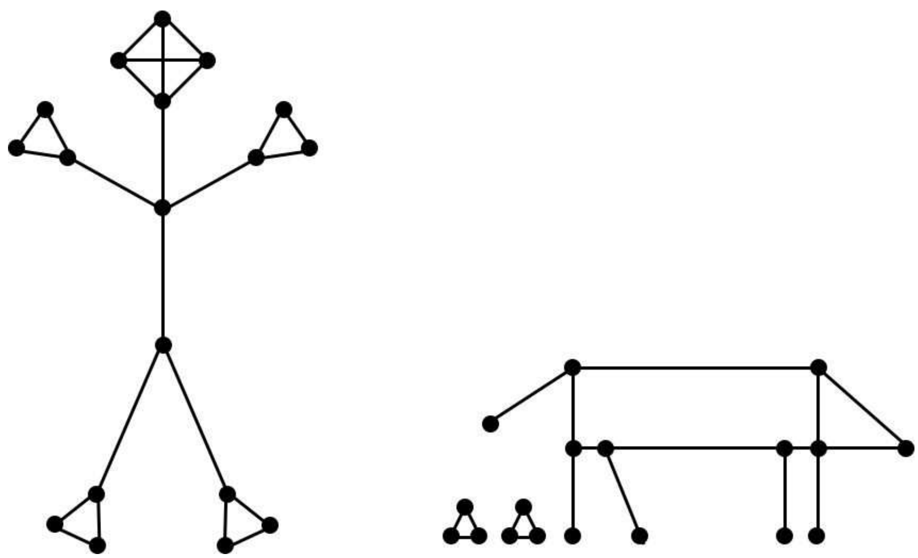

Figure 1: Mayer expansion diagrams for Clusterman and his dog Henry.

First of all, this is really disgusting and you should all be ashamed that you had anything to do with this problem.

(a) Clusterman's head gives a factor of 6 because the upper three vertices can be permuted among themselves in any of $3! = 6$ ways. Each of his hands gives a factor of 2 because each hand can be rotated by π about its corresponding arm. The arms themselves can be interchanged, by rotating his shoulders by π about his body axis (Clusterman finds this invigorating). Finally, the analysis for the hands and arms applies just as well to the feet and legs, so we conclude

$$s_\gamma = 6 \cdot \left(2^2 \cdot 2\right)^2 = 3 \cdot 2^7 = 384 \; .$$

Note that an arm cannot be exchanged with a leg, because the two lower vertices on Clusterman's torso are not equivalent. Plus, that would be a really mean thing to do to Clusterman.

(b) Henry himself has no symmetries. The little pieces each have $s_{\triangle} = 3!$, and moreover they can be exchanged, yielding another factor of 2. So the total symmetry factor for Henry plus disconnected pieces is $s_{\triangle\triangle} = 2! \cdot (3!)^2 = 72$. Were another little piece of the same…er…consistency to be found, the symmetry factor would be $s_{\triangle\triangle\triangle} = 3! \cdot (3!)^3 = 2^4 \cdot 3^4 = 1296$, since we get a factor of 3! from each of the $\triangle$ pieces, and a fourth factor of 3! from the permutations among the $\triangle$s.

(c) There are 18 vertices in Clusterman, hence he will first appear in B_{18}.

(2) Find an expression for the screened potential of a test charge Q in a two-dimensional system using an appropriate generalization of Debye-Hückel theory. The unscreened interparticle potential is $v(\boldsymbol{r},\boldsymbol{r}') = -2qq' \ln\big(|\boldsymbol{r}-\boldsymbol{r}'|/a\big)$, where a is a constant. Assume two species of charge, with $q = \pm e$, for the plasma. Show that at asymptotically large distances the test charged is perfectly screened.

Debye-Hückel theory gives

$$\nabla^2\phi = 8\pi e n_\infty \sinh\!\left(\frac{e\phi}{k_{\rm B}T}\right) - 4\pi\rho_{\rm ext} \ .$$

Assume $|e\phi| \ll k_{\rm B}T$, in which case

$$\nabla^2\phi = \kappa_{\rm D}^2\,\phi - 4\pi\rho_{\rm ext} \ ,$$

with $\kappa_{\rm D} = (8\pi n_\infty e^2/k_{\rm B}T)^{1/2}$. Note that e^2 has dimensions of energy in two space dimensions. Solving by Fourier transform, we have

$$\phi(\boldsymbol{r}) = \int\!\frac{d^2\!k}{(2\pi)^2}\,\frac{4\pi\hat\rho_{\rm ext}(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\boldsymbol{k}^2+\kappa_{\rm D}^2} \ .$$

With $\rho_{\rm ext} = Q\,\delta(\boldsymbol{r})$, we have

$$\phi(\boldsymbol{r}) = 2Q\,K_0(\kappa_{\rm D} r) \ ,$$

where $K_0(z)$ is the modified Bessel function of order zero. As $z \to 0$, one has $K_0(z) \sim -\ln z$, corresponding to an unscreened two-dimensional Coulomb potential. As $z \to \infty$, $K_0(z) \sim \sqrt{\frac{\pi}{2z}}\,e^{-z}$, and the potential is screened, with perfect screening overall. *I.e.* the charge in the screening cloud, integrated over all space, exactly compensates the external charge.

(3) Consider the equation of state

$$p\sqrt{v^2-b^2} = RT\,\exp\!\left(-\frac{a}{RTv^2}\right) .$$

(a) Find the critical point $(v_{\rm c}, T_{\rm c}, p_{\rm c})$.

(b) Defining $\bar{p} = p/p_c$, $\bar{v} = v/v_c$, and $\bar{T} = T/T_c$, write the equation of state in dimensionless form $\bar{p} = \bar{p}(\bar{v}, \bar{T})$.

(c) Expanding $\bar{p} = 1 + \pi$, $\bar{v} = 1 + \epsilon$, and $\bar{T} = 1 + t$, find $\epsilon_{\text{liq}}(t)$ and $\epsilon_{\text{gas}}(t)$ for $-1 \ll t < 0$.

(a) We write

$$p(T,v) = \frac{RT}{\sqrt{v^2 - b^2}}\, e^{-a/RTv^2} \qquad \Rightarrow \qquad \left(\frac{\partial p}{\partial v}\right)_T = \left(\frac{2a}{RTv^3} - \frac{v}{v^2 - b^2}\right) p \, .$$

Thus, setting $\left(\frac{\partial p}{\partial v}\right)_T = 0$ yields the equation

$$\frac{2a}{b^2 RT} = \frac{u^4}{u^2 - 1} \equiv \varphi(u) \, ,$$

where $u \equiv v/b$. Differentiating $\varphi(u)$, we find it has a unique minimum at $u^* = \sqrt{2}$, where $\varphi(u^*) = 4$. Thus,

$$T_c = \frac{a}{2b^2 R} \quad , \quad v_c = \sqrt{2}\, b \quad , \quad p_c = \frac{a}{2eb^2} \, .$$

(b) In terms of $\bar{p}$, $\bar{v}$, and $\bar{T}$, we have the universal equation of state

$$\bar{p} = \frac{\bar{T}}{\sqrt{2\bar{v}^2 - 1}} \exp\left(1 - \frac{1}{\bar{T}\bar{v}^2}\right) .$$

(c) With $\bar{p} = 1 + \pi$, $\bar{v} = 1 + \epsilon$, and $\bar{T} = 1 + t$, we have from eqn. 7.30 of the Lecture Notes,

$$\epsilon_{\mathsf{L,G}} = \mp \left(\frac{6\, \pi_{\epsilon t}}{\pi_{\epsilon\epsilon\epsilon}}\right)^{1/2} (-t)^{1/2} + \mathcal{O}(t) \, .$$

From `Mathematica` we find $\pi_{\epsilon t} = -2$ and $\pi_{\epsilon\epsilon\epsilon} = -16$, hence

$$\epsilon_{\mathsf{L,G}} = \mp \tfrac{\sqrt{3}}{2}\, (-t)^{1/2} + \mathcal{O}(t) \, .$$